



1509/1515

THE

Six First,

Together with the

Eleventh and Twelfth Books

OF

EUCLID'S ELEMENTS,

Demonstrated after a New, Plain, and Easie
METHOD.

By HENRY HILL, A. M.

Improbns. Virg.

Later omnia vicit

L O N D O N :

Printed by WILLIAM PEARSON, and Sold by R. and
J. BONWICKE, in St. Paul's-Church-Yard; F. FAYRAM, at
the South-Entrance of the Royal-Exchange; and B. MOTTE,
at the Middle-Temple-Gate, Fleet-Street. MDCCLXXVI.

THE ELEMENTS of EUCLID being
the most approv'd Book, for teach-
ing the *First Rudiments* of GEOMETRY; the
Author hereof, the Reverend Mr. HEN-
RY HILL, has taken Laudable Pains, to
render it as plain and easie to the Reader,
as the Nature of the Thing admits of.

April 8th 1724.



EDMUND HALLEY
Geom. Prof.

Printed by WILLIAM PEARSON and Sold by R. and
J. DODDING, in St. Pauls Church-Yard; H. KNEAULT,
the South-Entrance of the Royal Exchange; and H. MOSELEY,
at the Middle Temple Gate, in the Year MDCCLXXIV.



TO GEOMETRY we owe the mighty Progress, that
 has been already made in the
TO THE

Most High Puissant and Noble Prince

CHARLES,

Duke of Somerset, Marquiss and Earl of Hertford, Vis-
 count Beauchamp de Harche, Baron Seymour of Trow-
 bridge, Lord of the Honours of Cockermouth, Petworth,
 &c. Chancellour of the University of Cambridge, Knight
 of the most Noble Order of the Garter, and One of His
 Majesty's most Honourable Privy Council, &c. &c. &c.

May it please **YOUR GRACE.**

AMONG all the manifold Sciences, which
 have hitherto been the Study and Delight
 of the Learned World, there is none that has
 ever been esteemed of greater Extent, Use,
 and Profit, than that of **GEOMETRY.**

So vastly Advantageous, Useful, and Beneficial, is
 the Knowledge thereof to Mankind in General, that
 upon

Epistle Dedicatory.

upon it, as upon an Immoveable Pillar, is Super-
structed the most Spacious Science of the *Mathema-
ticks*; which so fully comprehends the Grounds and
Principles of all Arts, that there is hardly One which
stands not in need of its kind Assistance.

TO GEOMETRY we owe the mighty Progress, that
has been already made in *Statics*, *Hydrostatics*, *Op-
tics*, *Gnomonicks*, *Gunnery*, *Navigation*, and all other In-
feriour Branches of the *Mathematicks*; nor are we less
beholden to it for the Discovery of many New and
Surprising *Phanomina*; for the Explanation of the Laws
the Celestial Bodies observe in their Motions; and for
the Investigation of their true and Physical Causes.

THE ELEMENTS I here present YOUR GRACE, are
the *First Rudiments* of that Divine and Boundless Science,
which, like the great LUMINARY, the SUN, extends it
self from Pole to Pole, and scatters its Light through-
out all the Labyrinths of Humane Learning. They
are, MY LORD, the *First Principles* of that Sublime and
Noble Science, which has had the good Fortune not only
to be Universally receiv'd, but to be Admir'd also and
Encourag'd, by the greatest and most eminent Persons,
in all Ages and all Nations.

MAY it please YOUR GRACE, therefore to take the
following Sheets into your Patronage and Protection;
they being the *First Rudiments* of that Science, which
qualifies those that understand it for all Employ-
ments,

Epistle Dedicatory.

ments, and renders them Ingenuous in all Professions. May the Usefulness of the Subject recommend them to YOUR GRACES Favour, whatever may be wanting, either in the Work or the Author.

I should, MY LORD, offer at an Apology for presuming to throw in my poor Endeavours, among your Great and Important Affairs, did not the Work it self point out its Patron, and lay a more than ordinary Claim to YOUR GRACES Protection.

FOR, to whom, MY LORD, can I so properly Dedicate a Book calculated, and intended for the Use and Benefit of the World, as to a Person who is always promoting the general Good of Mankind, and continually employ'd in the Service of the Publick? To whom can I so well inscribe a Work of so much Intricacy, Labour, and Difficulty, as to a Person of YOUR GRACES Penetration, Candour, and Humanity? To whom can I with so much Justice send a Treatise of the *First Principles* of almost every Art, as to One who is well acquainted with every Science, and adorn'd with Learning of every Sort?

YOU, MY LORD, are the most proper Person, to introduce the *First Rudiments* of Arts and Sciences to the Publick, and to recommend them to the Learned World; Learning being the Employment and Entertainment of your noble Mind. Such is your Affection for Arts, that you countenance and encourage
a every

Epistle Dedicatory.

every Artist; And so great is your Reputation for Learning, that you have the Honour to preside over one of the most Learned Bodies in the Universe.

MUCH more, MY LORD, might be alledg'd, for laying this poor Performance at YOUR GRACES Feet, which is, I confess, infinitely unworthy of the Acceptance of so great a Person, and the Perusal of so Judicious an Eye; but fearing least I should offend your Modesty, and remembring how valuable every Moment must be to one of your High Birth, Rank, and Quality; I am the rather determin'd to wave the Prosecution, and to rely upon YOUR GRACES Candour and Condescension.

MAY the PRINCE of GEOMETRICIANS, the OMNIPOTENT CREATOR of the Universe, long preserve YOUR GRACE, and may you long enjoy that Affluence, which, like the SUN it self, affords Warmth and Life to all about you.

THIS, MY LORD, is the sincere Wish, and hearty Prayer of him, who with the utmost Submission and Respect, craves leave to Subscribe himself,

YOUR GRACES,

Most Obedient,



And most Humble Servant,

Henry Hill.



THE PREFACE.



IS expected of those who appear in Publick, that they give the World a short Account, of the Usefulness of the Subjects upon which they Treat; of the chief Motives which induc'd them to Write; and of the Methods they have taken and made choice of in Writing.

THIS is that which is required of all Authors in General; but more especially of those, who undertake Subjects, which are not only Intricate and Difficult in themselves, but which have been often handled, and that by Men of the greatest Genius's, Judgments, and Abilities.

THE Contents then of the following Sheets, being the First Rudiments of Geometry, the Usefulness of the Subject, will, I presume, appear, from that of Geometry itself; there being no Science whatsoever, that can afford greater Delight or Entertainment to the Mind; that is so Extensive and Unbounded in its Use and Application; or that can lay a more just Claim, either to the Invention or the Improvement of all useful Mechanick Arts.

THE chief Motive which induc'd me to Write, was the Usefulness of the Subject upon which I Treat; it being the infinite Advantages which accrue from Geometry to Mankind in General, that led me to compose the ensuing Treatise, and to set out every Demonstration in such a regular, plain, and easie Manner, as might neither burthen the Memory, nor perplex the Mind; but that the Young Student might proceed with Ease and Cheerfulness, in the First Rudiments of that great and spacious Science, which is the Basis and Foundation of all Parts and Branches of

The PREFACE.

of the Mathematicks, and which intitules those that understand it to such Penetration and Exactness, as fits and prepares them for all Employments.

THE Method I have taken in Writing, is entirely new; wholly different from any that has hitherto been made choice of by any of the Interpreters of EUCLID, and such as does not so much require Intent and severe Thinking, as a bare and easie Inspection.

THE letter to avoid Obscurity and Confusion, and that painful Attention, which we have a general Aversion to, and which disturbs and distracts the Mind; I have set each, and every Demonstration out in Lines, or Steps; so that the several Medium's that are necessary, in Order to investigate the Truth of many Propositions, might no ways puzzle or perplex the Young Geometrician; but that he might have a clear and perfect Notion and Idea, of the most Intricate Demonstrations in the ELEMENTS of GEOMETRY, and proceed with Courage and Delight, in quest of the First Principles of real and true Philosophy.

BESIDES that of Steps, which runs throughout the Book, I have (in order to facilitate the Work) had recourse sometimes to one Method, sometimes to another, according as I judg'd this or that of them, fittest to Demonstrate the Thing in Hand; but having chiefly, and for the most Part, made use of Algebra, where I thought it convenient to deviate from the Original, I have, for that Reason, prefixt as much of Literal Arithmetick, as is any where required in the following Demonstrations.

WHAT Opinion the World may have of these my Endeavours, after so many Ingenuous and Elaborate Performances on the same Subject, Time must teach me; all that I have to request, is, that the following Sheets may be impartially perus'd, and not too rigorously censur'd, but that the Candid Reader would excuse and pass over any slight Error, that may possibly occur in the ensuing Treatise.



The



*The Explanation of the Signs, Characters, and Abbreviations of Words,
used in this B O O K.*

$+$	More, or to be added.
$-$	Less, or to be subtracted.
$\times$	Multiplied by.
$\div$	Divided by.
$\frac{A}{B}$	A divided by B.
$1 + 2$	The first Step added to the second Step.
$1 + 2 AB$	The first Step added to the Quantity $2 AB$.
$1 - 2$	The first Step wanting the second Step.
$1 - 2 AB$	The first Step wanting the Quantity $2 AB$.
1×2	The first Step multiplied by the second Step.
1×2	The first Step multiplied by the absolute Number 2 .
$1 \div 2$	The first Step divided by the second Step.
$1 \div 2$	The first Step divided by the absolute Number 2 .
$=$	Equality, or equal to.
$\sqsupset$	Majority, or greater than.
$\sqsubset$	Minority, or less than.
$\parallel$	Parallel to.
$\therefore$	Therefore.
$e.$	From.
$\angle$	Angle.
$\angle'$	Angles.
$r\angle$	Right-angle.
$r\angle'$	Right-angles.
$\triangle$	Triangle.
$\triangle'$	Triangles.
$\perp$	Perpendicular, and Perpendicular to.
$\perp'$	Perpendiculars.

□	Parallelogram.
□	Parallelograms.
⊙	Involution, or the Raising or Producing of Powers, from any proposed Root.
⊙	Evolution, or the Extracting of Roots from any given Power.
⊙	The Squaring of any Quantity, or Involving of it to the second Power.
⊙	The Extracting of the Square Root of any Quantity.
A	The Cube of A.
A ³ B	The Cube of A B.
q	Square.
q	Square.
√	Square Root.
A:B::C:D.	As A is to B, so is C to D.
Ax.	Axiom.
Post.	Postulate, or Petition.
Def.	Definition.
Const.	Construction.
Sub.	Substitution, or the placing of one Quantity in the Room of another, which is equal to it.
Hyp.	Hypothesis, or Supposition.



A LIST of the Subscribers to, and Encouragers of this
WORK.

A

The Right Honourable the Countess Dowager of *Aylesford*.
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Anonymus.

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Y

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ry of *Sarum.*

THE





THE SIX FIRST,

Together with the

ELEVENTH and TWELFTH BOOKS

EUCLID'S ELEMENTS,

Demonstrated after a New, Plain, and Easy

METHOD.

BOOK I.

DEFINITIONS.

Geometrical Point, is that which has no Parts, or at least is considered as such, and which of Consequence is indivisible.



B

A

- 2 A *Line*, whether *Straight* or *Crooked*, is a Length without either Breadth or Thickness, and is conceiv'd to be generated by the Motion of a Point; as AB, CD.

Because a *Line* hath Length, it is called a Quantity of one Dimension, and is supposed to consist of an infinite Number of Points.



- 3 The *Ends*, or *Limits* of a *Line*, are Points.

- 4 A *Right-line*, is that whereof the Points are equally placed between its Two *Extremities*; as AB.

Whence it follows, that a *Curve-line*, is that which has not all its Points placed equally between its two *Extremities*; because some are elevated above, and some subside below others; as CD.

- 5 A *Superficies* or *Surface*, is that which hath Length and Breadth without Thickness, and is conceiv'd to be generated by the Motion of a *Line*, as a *Line* is by the Motion of a Point.

Thus, if AB, were equally mov'd along the Line AC, to CD; it would form the *Superficies* or *Surface* ABCD.



Because a *Superficies* or *Surface* hath Length and Breadth, it is called a Quantity of two Dimensions, and is supposed to consist of an infinite Number of *Lines*.

The

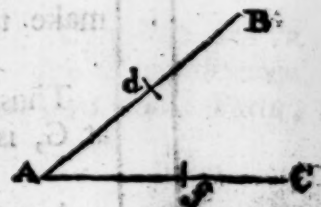


6 The *Extremities*, or *Ends* of a Superficies or Surface, are *Lines*.

7 A *Plane-Surface* or *Plane*, is that which hath all its *Right-lines* equally placed between its *Extremities*; so that one does not rise higher, or subside lower than the other; as *ABCD*.

8 A *Plane-Angle*, is the *Inclination* of two *Lines* the one to the other, the one touching the other in the same *Plane*, yet not lying in the same *Straight Line*; as *BAC*.

And according as such *Lines* are inclin'd to each other, that is, stand nearer or further off each other, the *Angle* is said to be lesser or greater, whether the *Lines* which include it be short or long.



Thus, the *Lines* *Ad*, *Af*, include the same *Angle*, as the *Lines* *AB*, *AC*; notwithstanding that *AB* is longer than *Ad*, and *AC* than *Af*.

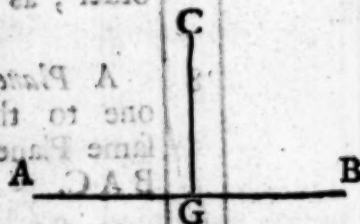
9 All *Angles*, whether *Right*, *Obtuse*, or *Acute*, which are included between *Right-lines*, are called *Right-lin'd Angles*.

Be 2.

When

10 When a Right-line CG , standeth on a Right-line AB , and maketh the Angles CGA , CGB ; on either Side thereof, equal one to the other, then both those equal Angles are *Right-Angles* and the Right-line CG , which standeth on the other AB , is term'd a *Perpendicular* to that other AB , whereon it standeth.

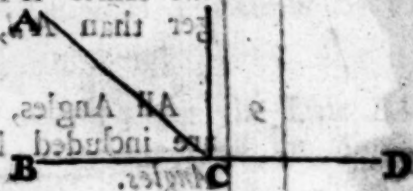
Note. When several Angles meet at the same Point, as at G , each particular Angle is described by three Letters; whereof the middle Letter sheweth the angular Point, and the two other Letters, the Lines that make the Angle.



Thus, the Angle which the Right-lines CG , AG make at G , is called CGA , or AGC .

11 An *Obtuse-Angle* is that which is greater than a Right-Angle; as ACD .

12 An *Acute-Angle* is that which is less than a Right-Angle; as ACB .



13 A *Limit*, or *Term*, is the End of any Thing.

14 A *Figure* is any Space or Quantity of two or three Dimensions, comprehended under, or bounded every Way by one or more Terms.

- 15 A *Circle* is a plane Figure, terminated by a Boundary of one Line only, which is called its *Periphery* or *Circumference*, as ABCDA, within which is a Point, as E, called the *Center*; from which all the Right-lines, *Semidiameters*, or *Radij*, EA, EB, EC, &c. drawn to the *Circumference*, are equal to one another.

- 16 The *Center* therefore of a *Circle* is a Point within its *Circumference*, from which all Right-lines drawn to that *Circumference*, are equal among themselves; as E.



- 17 The *Diameter* of a *Circle*, is any Right-line drawn through the *Center*, and terminated by the *Circumference* on each Side, dividing the *Circle* into two equal Parts; as AC.

- 18 A *Semicircle*, is a Figure which is contained under the *Diameter*; and under that Part of the *Circumference* which is cut off by the *Diameter*; as ABC.

- 19 *Right-lin'd Figures*, are such as are contained under Right-lines.

- 20 *Three-sided Figures*, or *Triangles*, are such as are contain'd under three Right-lines; as A.

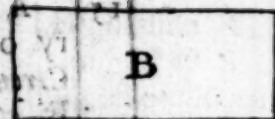


C

Four

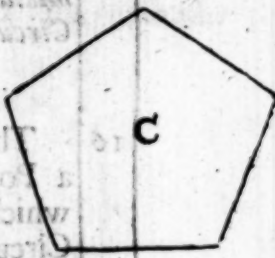
21

Four-sided, or Quadrilateral Figures, are such as are contained under Four Right-lines; as B.



22

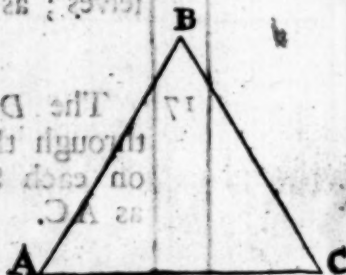
Many-sided Figures, or Polygons, are such as are contain'd under more Right-lines than four; as C.



23

An *Equilateral Triangle*, is that which has all its Three Sides equal; as A.B.C.

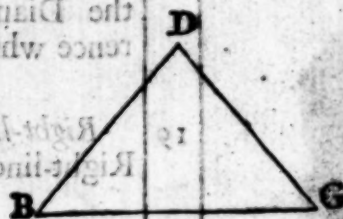
That is, $AB = BC = AC$.



24

An *Isoceles Triangle*, is that which hath only Two of its Sides equal; as D.B.G.

That is, $DB = DG$; but the Third B.G., may be either greater or less, as Occasion requires.



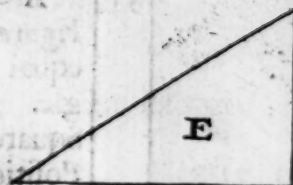
25

A *Scalenous Triangle*, is that which hath all its Three Sides unequal; as D.



A

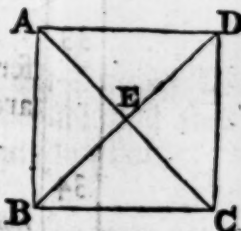
- 26 A *Right-angled Triangle*, is that which hath one *Right-angle*; as E.



- 27 An *Obtuse-angled Triangle*, is that which hath One of its Angles *Obtuse*; as D.

- 28 An *Acute-angled Triangle*, is that which hath all its Angles *Acute*; as ABC.

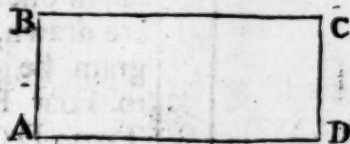
- 29 A *Square*, is a *Quadrilateral Figure* contain'd under Four equal Sides, all perpendicular to one another; as ABCD.



That is, when $AB = BC = CD = AD$, and the Angles A, B, C, D, are equal, then it is usually called a *Geometrical Square*.

A *Right-line* drawn from any Angle of a *Square*, to the opposite One, as AC, DB, is called its *Diagonal*; and the Point where Two such *Diagonals* intersect each other, is called its *Center*; as E.

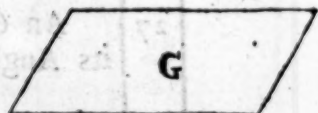
- 30 A *Rectangle*, or *Right-angled Parallelogram*, (often called an *Oblong*, or *Long Square*;) is a Figure that hath Four *Right-Angles*, and its Two opposite Sides equal, viz. $BC = AD$, and $BA = CD$.



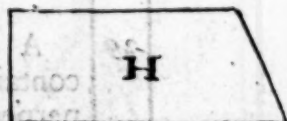
- 31 A *Rhombus*, or *Diamond-like Figure*, is that which hath Four equal Sides, but no Right-Angle. That is, a *Rhombus* is a Square moved out of its right Position; as F.



- 32 A *Rhomboid*, or *Diamond-like Figure*, is that whose opposite Sides, and opposite Angles are equal; but has neither equal nor Right-Angles; as G.



- 33 All Four-sided Figures, which differ from those before mentioned, are called *Trapezia*, or *Tables*; as H.

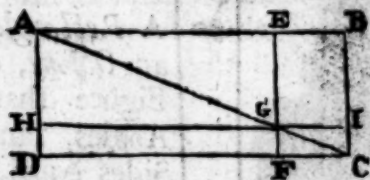


- 34 *Parallel*, or *Equidistant Right-lines*, are such as being in the same Superficies, if infinitely produced, would never meet; as A B and B.



- 35 A *Parallelogram*, is a Quadrilateral Figure, whose opposite Sides are Parallel, or Equidistant; as G.

- 36 In a Parallelogram A B C D, when a Diameter A C, and two Lines E F, H I, parallel to the Sides, cutting the Diameter in one and the same Point G, are drawn, so that the Parallelogram be divided by them into Four Parallelograms; those Two, D G, G B, thro' which the Diameter passeth not, are called *Complements*; and the other Two H E, F I, through which the Diameter passeth, the *Parallelograms standing about the Diameter*.



37 A *Problem*, is, when something is proposed to be done or effected.

38 A *Theorem* is, when something is to be demonstrated.

39 By *Demonstration* is understood the highest Degree of Proof that human Reason is capable of attaining to, by a Train of Arguments, deduced or drawn from such plain Axioms, and other self-evident Truths, as cannot be denied by any One that considers them.

40 A *Corollary*, is a *Consequent*, or some consequent Truth gained from a preceding Demonstration.

41 A *Lemma*, is the Demonstration of some Premiss laid down, or proposed as a Preparative to obviate and shorten the Proof of the Theorem under Consideration.

42 A *Scholion*, is a brief Commentary, or Observation made upon some precedent Discourse.

POSTULATES OR PETITIONS.

1 That a *Right-line* may be drawn from any one given *Point* to another.

2 That a *Right-line* may be *Produced*, *Increased*, or made *Longer*, from either of its *Ends*.

3 That upon any given *Point* (or *Centre*) and with any given *Distance*, (*viz.* with any *Radius*) a *Circle* may be described.

A X I O M S

1 Quantities equal to the same Third, are also equal one to the other.

As $A = B = C$. Therefore $A = C$. Or therefore A, B, C, are equal the one to the other.

2 Any Quantity that is less or greater than either of two equal ones, is also less or greater than the other.

As $C < \text{or } > A$, and $A = B$. Therefore $C < \text{or } > B$.

3 If of two equal Quantities the one is greater or less than a Third, the other shall be also greater or less than that Third.

As $A = B$, and $A < \text{or } > C$. Therefore $B < \text{or } > C$.

4 A Quantity is equal to another Quantity, when it is neither greater nor less than that Quantity.

As $A = B$, when it is neither $<$ nor $> B$.

5 A Quantity is greater than another Quantity, when it is neither equal nor less.

As $A < B$, when it is neither $=$ nor $> B$.

6 A Quantity is less than another Quantity, when it is neither equal nor greater.

As $A > B$, when it is neither $=$ nor $< B$.

7 Quantities that are double each of the same Quantity, or of equal Quantities, are equal among themselves.

As $A = 2 C$, and $B = 2 C$. Therefore $A = B$.

Understand the same of Triple, Quadruple, &c.

Quanti-

8 Quantities that are each one half of the same Quantity, or of equal Quantities, are equal among themselves.

As $A = \frac{1}{2} C$, and $B = \frac{1}{2} C$. Therefore $A = B$.

Understand the same of Subtriple, Subquadruple, &c.

9 If the Roots of any two Quantities are equal, their Squares are also equal.

As $A = B$. Therefore $AA = BB$.

10 If the Squares of any two Quantities are equal, their Square Roots are also equal.

As $AA = BB$. Therefore $A = B$.

11 If to equal Quantities you add equal Quantities, their Sums will be equal.

Thus $6 = 6$. And $6 + 3 = 6 + 3 = 9$.

12 If from equal Quantities, you subtract equal Quantities, their Remainders will be equal.

Thus, $6 = 6$. And $6 - 3 = 6 - 3 = 3$.

13 If equal Quantities, be multiplied by equal Quantities, their Products will be equal.

Thus, $6 = 6$ and $6 \times 3 = 6 \times 3 = 18$.

14 If equal Quantities, be divided by equal Quantities, their Quotients will be equal.

Thus, $6 = 6$. And $6 \div 3 = 6 \div 3 = 2$.

15 If to unequal Quantities, you add equal Quantities, their Sums will be unequal.

Thus, $8 \neq 4$. And $8 + 2 = 10$, is $\neq 4 + 2 = 6$.

16 If from unequal Quantities, you subtract equal Quantities, their Remainders will be unequal.

Thus, $8 - 4 = 4$. And $8 - 2 = 6$, is $4 - 2 = 2$.

17 Every whole is greater than its Part.

Thus, the whole Line AB is greater than its Part AC .
The same is to be understood of Superficies and Solids.

18 Every whole is equal to all its Parts taken together.

Thus, the whole Line AB is equal to all its Parts $AC + CB$.

The same is also true in Superficies and Solids.

19 Quantities which every Way agree, are equal.

That is, if two Lines being placed one upon the other, do so agree, as that all the Parts of the one correspond exactly to all the Parts of the other; so that neither surpasses (or is greater, or less than) the other, those two Lines are equal.

The same may be said of two Angles, of two Surfaces, or of two Solids, when one being placed upon the other, or supposed to penetrate the other, neither of them surpasses the other.

20 Two Right-lines cannot comprehend, (or include) space, or constitute a Figure.

For it is evident, that two Right-lines meeting one another, can only make an Angle, which is not a Figure.

21 All Right-Angles are equal to one another.

This follows from the Definition of a Perpendicular, which supposes, that it makes on the Line on which it falls two equal Angles, which we call Right-Angles.

If

22 If a Right-line B A, falling on two Right-lines A D, C B, make the internal Angles on the same Side, B A D, A B C, less than two Right-angles, those two Right-lines produced shall meet on that Side, where the Angles are less than two Right-angles.

Of Tracing the Steps us'd in bringing Quantities to an Equation.

The Method of Tracing the Steps, us'd in bringing Quantities to an Equation, is best perform'd by Registering the several Operations with Figures and Signs, placed in the Margin of the Work, according as the several Operations require.

For Instance, If it were requir'd to set down, and register the Sum of the two Quantities A, and B, the Work will stand,

Thus	1	A	First set down the propos'd Quantities, A and
	2	B	B, over-against the Figures 1, 2, in the small
	3	<u>A + B</u>	Column, which are here call'd Steps, and against
1 + 2			3, (the third Step) set down their Sum, viz.
			A + B: Then against that third Step set down
			1 + 2 in the Margin; which denotes that the Quanti-
			ties against the first and second Steps are added together,
			and that those in the third Step are their Sum.

To

To illustrate this in Numbers, suppose $A = 6$, and $B = 3$.

Then it will be,

Thus	1	$A = 6$
	2	$B = 3$
$A + B$	3	$6 + 3 = 9$

$A + B = 6 + 3 = 9$. the Sum of 6 and 3.

Again, if it were requir'd to set down their Difference.
Then it will be,

Thus	1	$A = 6$
	2	$B = 3$
$A - B$	3	$6 - 3 = 3$

$A - B = 6 - 3 = 3$. the Difference between 6 and 3.

Moreover, if it were requir'd to set down their Product.
Then it will be,

Thus	1	$A = 6$
	2	$B = 3$
$A \times B$	3	$6 \times 3 = 18$

$A \times B = 6 \times 3 = 18$. the Product of 6 into 3.

Lastly, if it were requir'd to set down their Quotient.
Then it will be,

Thus	1	$A = 6$
	2	$B = 3$
$A \div B$	3	$\frac{6}{3} = 2$

$\frac{A}{B} = \frac{6}{3} = 2$. the Quotient of 6 by 3.

Addition of Algebraick Integers.

CASE I.

If the Quantities are alike, and have like Signs; add the Co-efficients or prefix'd Numbers together; and to their Sum adjoyn the Quantities with the same Sign.

EXAM

EXAMPLES.

Thus	1	$+1A$	$-1A$	$5B$	$-7BC$	$3A+12$	$YY+2X-2$
	2	$+1A$	$-1A$	$8B$	$-10BC$	$2A+6$	$YY+3X-5$
$1+2$	3	$+2A$	$-2A$	$13B$	$-17BC$	$5A+18$	$2YY+5X-7$

The Reason of these Additions will appear, if we consider that if to $1A$, which may represent one Crown, be added $1A$, or one Crown, the Sum will be $2A$, or two Crowns; also if to $-1A$, which may represent the Want or Debt of one Crown, be added $-1A$, or the Want or Debt of another Crown, the Sum will be $-2A$, or the Want or Debt of two Crowns.

CASE II.

If the Quantities are alike, and have unlike signs; subtract the Co-efficients from each other, and to their Difference joyn the Quantities with the Sign of the greater Quantity.

EXAMPLES.

Thus	1	A	$-5A$	$3BC$	$7A-5B$	$-7ABD+15$
	2	$-A$	$+3A$	$-2BC$	$-3A+7B$	$+9ABD-12$
$1+2$	3	0	$-2A$	BC	$4A+2B$	$2ABD+3$

The Reason of the Operations in this Case will be easily understood, if we consider that to add a Negative, is to take away a Positive; therefore to connect a Negative and a Positive, is to make the one to destroy the other.

Thus,

Thus, If A has 600*l*. and owes 400*l*. 'tis plain that the Sum, or his worth is but 200*l*.

And if A has 600*l*. and owes 900*l*. then the Worth is — 300*l*. or 300*l* worse than Nothing.

CASE III.

When the Quantities are unlike, set them all down without altering their Signs.

EXAMPLES.

Thus	1	4 M		A		6C + 7DC		5X + Y + Z
	2	5 H		— B		4A		B + C — M
1 + 2	3	4M + 5H		A — B		6C + 7DC + 4A		5X + Y + Z + B + C — M

The Reason of the Operations in this Case will appear, if we suppose M to represent Miles and H Hours; for then 4 M + 5 H, that is, 4 Miles added to 5 Hours, will make neither 9 Miles nor 9 Hours.

Subduction of Algebraick Integers.

General Rule.

Change all the Signs of the Subtrahend; (*viz.* of those Quantities which are to be subtracted) or suppose them in your Mind to be chang'd. Then add all the Quantities together, as before in Addition, and their Sum will be the true Remainder or Difference requir'd.

EXAM-

EXAMPLES.

Thus	1	+ A	- B	Y	- X	3 A A - 5 A + 7
	2	+ A	- B	- Y	X	A A + A - 2
1 - 2	3	A - A	- B + B	Y + Y	- X - X	2 A A - 6 A + 9
		0	0	2 Y	- 2 X	

The Reason of this *General Rule* will appear, if we consider.

1st. That if a Positive be taken from its Positive, that is, + A from + A, the Remainder will be 0; since to take away the Affirmation of any Quantity A, is to make it Negative, or - A; so that the one will destroy the other.

2^{dly}. That if a Negative be taken from its Negative, that is, - B from - B, the Remainder will be 0; since to take away the Negation of any Quantity B, is to make it Affirmative, or + B; so that the one will destroy the other.

3^{dly}. That if a Negative be taken from its Positive, that is, - Y from + Y, the Remainder will be double of that Positive, viz. 2 Y; since to subtract or take away a Privation or Negative, is to add that very Thing the Privation of which you take away.

4^{thly}. That if a Positive be taken from its Privative, that is, + X from - X, the Remainder will be double the Privative One, viz. - 2 X; since by taking away a Positive One, there necessarily arises a new Privation, which will double that you had before.

Multipli.

Multiplication of Algebraick Integers.

CASE I.

When the Quantities have like Signs, and no Co-efficients, set or join them together; and prefix the Sign + before them, and that will be their Product.

EXAMPLES.

Thus	1	A	- A	A + B	- A - B	A + R
	2	B	- B	D	- D	R
1 x 2	3	AB	+ AB	AD + DB	+ AD + DB	AR + RR

CASE II.

If there be Co-efficients; multiply them, and to their Product adjoin the Quantities set together as before.

EXAMPLES.

Thus	1	5 A	- 6 D	3 A + 2 B	2 A + 5 R	A + B
	2	3 B	- 7 B	6	3 R	8 B
1 x 2	3	15 AB	+ 42 BD	18 A + 12 B	6 AR + 15 RR	8 AB + 8 BB

The Reason of these Operations will appear, if we consider that to multiply an Affirmative by an Affirmative Quantity, is so often to repeat the Affirmation of it; and consequently + into + will give + in the Product.

But

But to multiply a Negative by a Negative Quantity, is so many Times to deny that Negation.

And so many Times to deny the Negation of a Thing, is so many Times to affirm that Thing.

Therefore — into — will give + in the Product.

CASE III.

When Quantities have unlike Signs, join them and the Product of their Co-efficients together, (as before) but prefix the Sign — before them.

EXAMPLES.

Thus

1	A	— 6D	— 4A — 7B	— A — B	A + B
2	— B	+ 7B	3F	C	— C
3	— A B	— 42 BD	— 12 AF — 21 BF	— AC — BC	— AC — BC

The Reason of these Multiplications may be easily understood, if we consider that to multiply an Affirmative by a Negative, is so often to repeat the Negation of it; and consequently — into +, or + into —, will give — in the Product.

Note: If the Multiplier consists of several Terms, then every one of those Terms must be multiplied into all the Terms of the Multiplicand: And the Sum of those particular Products, will be the Product requir'd.

F 2

EXAM-

EXAMPLES.

Thus	1	$A + B - D$	$7B + 5D$
	2	$A - B$	$3A - 5F$
$1 \times A$	3	$AA + AB - AD$	$21AB + 15AD$
$1 \times -B$	4	$-AB - BB + BD$	$-35BF - 25DF$
$3 + 4$	5	$AA - AD - BB + BD$	$21AB + 15AD - 35BF - 25DF$

Division of Algebraick Integers.

CASE I.

When the Quantities in the Dividend, have like Signs to those in the Divisor, and no Co-efficients in either; cast off or expunge all the Quantities in the Dividend, that are like those in the Divisor; and set down the other Quantities with the Sign + for the Quantities requir'd.

EXAMPLES.

Thus	1	AB	$-AB$	$AD + BD$	$-AD - BD$	$CXXXZYYY$
	2	B	$-B$	D	$-D$	Y
$3 \div 2$	3	A	$+A$	$A+B$	$+A+B$	$CXXXZYY$

CASE II.

When the Quantities in the Dividend, have unlike Signs to those in the Divisor; then set down the Quotient Quantities found as before, with the Sign - before them.

EXAM

EXAMPLES.

Thus	1	+AB	-AB-BD	ABC+BCD	XX+ZX
	2	-B	+B	-BC	-X
1 ÷ 2	3	-A	-A-D	-A-D	-X-Z

CASE III.

If the Quantities in the Dividend and Divisor, have Co-efficients; divide the Numbers (as in common *Arithmetick*) and to their Quotient adjoin the Quotient Quantities.

EXAMPLES.

Thus	1	15AB	9BC	42DB	12AF-21BF	AB
	2	3B	-9BC	-7B	3F	AB
1 ÷ 2	3	5A	-1	-6D	4A - 7B	+1

CASE IV.

When the Quantities in the Divisor cannot be exactly found in the Dividend; then set them both down like a Vulgar Fraction.

EXAM-

EXAMPLES.

Thus	1	A	6BC	5B+AA	8ADC	XX+YY+Z
	2	B	3D	5D+5C	4ABC	3B+2D
	3	A	6BC	5B+AA	8ADC	XX+YY+Z
$x \div 2$		B	3D	5D+5C	4ABC	3B+2D

Division being the Converse of Multiplication, the Truth of these Operations will appear, by multiplying each Divisor into its respective Quotient, which (according to the Rules laid down in Multiplication) will produce the Dividend. As in common *Arithmetick*.

Involution of Quantities.

A Quantity multiplied into it self any Number of Times, is said to be Involv'd; the Products arising are called Powers, and the Quantity so multiplied, a Root.

$$\text{Thus } \begin{cases} A \times A = AA \text{ or } A^2. \\ A \times A \times A = AAA \text{ or } A^3. \text{ \&c.} \end{cases}$$

Here A is the Root or first Power; A^2 , A^3 , &c. The Second, Third, &c. Powers of it.

Involution of Compound Quantities is perform'd in the same Manner, due regard being had to their Signs and Co-efficients, if there be any.

For Instance, Suppose $A + B$ were given to be Involv'd to the second Power.

$$\begin{array}{rcl}
 & & A + B \\
 1 & \times & A \\
 \hline
 1 & \times & B \\
 2 & & AA + AB \\
 3 & & + AB + BB \\
 \hline
 2 + 3 & & 4 \quad AA + 2AB + BB \text{ the Square of } A + B.
 \end{array}$$

Note. I would advise the Learner, to make himself thoroughly acquainted with the preceding *Definitions*, *Postulates*, *Axioms* and *Algebraick Arithmetick*; as also with the *Definitions* prefixt to the subsequent Books.

PROPOSITION I

Upon a finite Right-line AB, to describe an Equilateral Triangle ABC.

PREPARATION.

From the Centers A and B, at the Distance of AB, or BA, describe (*post. 3.*) two Circles cutting each other in the Point C; from whence draw (*post. 1.*) two Right-lines CA, CB.

I say, $AB = BC = AC$.

DEMONSTRATION.



1 $AB = BC$. *Def. 16.*

Also 2 $AB = AC$. *Def. 16.*

c. 1, 2. 3 $AB = BC = AC$. *Ax. 1. Q.E.F. & D.*

G 2

PROP.

PROP. II.

At a Point given A, to make a Right-line AF, equal to a Right-line given BC.

PREPARATION.

Draw (*post. 1.*) the Right-line AB, and upon it describe (*1.*) an Equilateral Triangle ABD. With the Radius BC, describe (*post. 3.*) the Arch of a Circle ICEK, and produce (*post. 2.*) DB to the Point E, in that Arch. With the Radius DE, describe (*post. 3.*) the Arch of a Circle GEFH, and produce (*post. 2.*) DA to the Point F, in that Arch.

I say, $AF = BC$.

DEMONSTRATION.



- | | | |
|-------|---|---|
| | 1 | $DF = DE$. <i>Def. 16.</i> |
| | 2 | $DA = DB$. <i>Const.</i> |
| And | 3 | $DF - DA = DE - DB$. <i>Ax. 12.</i> |
| — 2 | 4 | $AF = BE$. |
| i. e. | 5 | $BC = BE$. <i>Def. 16.</i> |
| Also | 6 | $AF = BC$. <i>Ax. 1. Q.E.F. & D.</i> |

PROP. III.

Two unequal Right-lines being given, to cut off from the greater AB, a Part AE equal to the lesser CD.

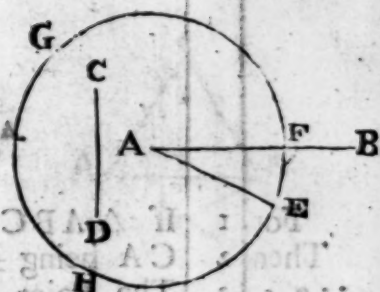
PRE-

PREPARATION.

Draw (1.) from the Point A, the Line $AE = CD$, and with the Radius AE , describe (*post.* 3.) the Circle $GFEH$.

I say, $AB = CD$.

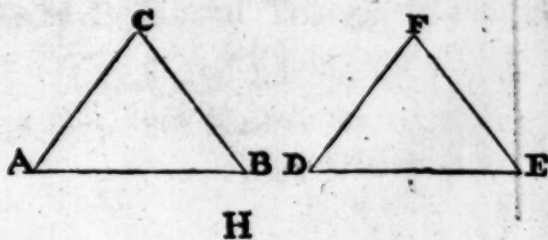
DEMONSTRATION.



- 1 $AF = AE$. Def. 16.
 Also 2 $CD = AE$. Const.
 ∴ c. 1. 2. 3 $AB = CD$. Ax. 1. Q.E.F. & D.

PROP. IV.

If two Triangles ABC , DEF , have two Sides of the one CA , CB , equal to two Sides of the other FD , FE , each to its correspondent Side (that is, $CA = FD$, and $CB = FE$) and have the Angle C , equal to the Angle F , comprehended between those equal Sides; they shall have the Base AB , equal to the Base DE ; and the Triangle ABC , shall be equal to the Triangle DEF ; and the remaining Angles A, B , shall be equal to the remaining Angles D, E , each to each, under which the equal Sides are subtended.



That

That is, $AB = DE$, $\triangle ABC = \triangle DEF$, $\angle A = \angle D$,
 $\angle B = \angle E$.

DEMONSTRATION.



- For 1 If $\triangle ABC$ be placed on $\triangle DEF$,
 Then 2 CA being $= FD$. Hyp.
 $\therefore$ e. 2. 3 The Point A will fall on D , the Point C on F .
 Also 4 $\angle C$ being $= \angle F$, $CB = FE$. Hyp.
 $\therefore$ e. 4. 5 CB will fall on FE , and B on E .
 $\therefore$ e. 3. 5. 6 The Points A, C, B , will fall on D, F, E .
 $\therefore$ e. 6. 7 AB will fall on DE . Ax. 20.
 $\therefore$ 8 $AB = DE$, $\triangle ABC = \triangle DEF$, $\angle A = \angle D$,
 $\angle B = \angle E$. Ax. 19. Q. E. D.

PROP. V.

The Angles CAB, CBA , at the Base of an Isosceles Triangle ABC , are equal one to the other: And if the equal Sides CA, CB , be produced, the Angles ABE, BAD , under the Base shall be equal one to the other.

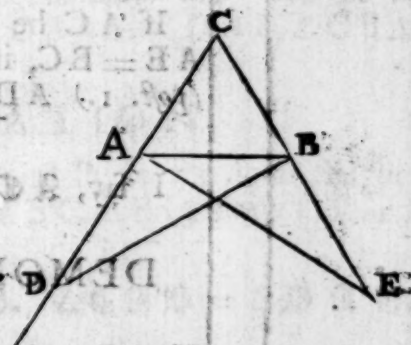
PRE-

PREPARATION.

Take (3.) $CD = CE$; and (post. 1.) join BD , and AE

I say, $\angle CAB = \angle CBA$. $\angle ABC = \angle BAD$.

DEMONSTRATION.



- | | | | |
|--------------------------|----|---|--|
| | 1 | $CD = CE$. <i>Const.</i> | |
| And | 2 | $CA = CB$. <i>Const.</i> | |
| 1 — 2 | 3 | $CD - CA = CE - CB$. <i>Ax. 12.</i> | |
| i. e. | 4 | $AD = BE$. | |
| Again | 5 | $\angle C$ is common. | |
| $\therefore$ e. 2. 1. 5. | 6 | $CA = CB$, $CE = CD$, $\angle C$ is common. | |
| $\therefore$ e. 6. | 7 | $AE = BD$, $\angle E = \angle D$, $\angle CAE = \angle CBD$. 4. | |
| Also e. 4. 7. | 8 | $BE = AD$, $AE = BD$, $\angle E = \angle D$. | |
| $\therefore$ e. 8. | 9 | $\angle ABE = \angle BAD$. 4. | |
| Also e. 8. | 10 | $\angle BAE = \angle ABD$. 4. | |
| Lastly e. 7. | 11 | $\angle CAE = \angle CBD$. | |
| 11 — 10. | 12 | $\angle CAE - \angle BAE = \angle CBD - \angle ABD$. <i>Ax. 12.</i> | |
| i. e. | 13 | $\angle CAB = \angle CBA$. | |
| $\therefore$ e. 13. 9. | 14 | $\angle CAB = \angle CBA$. $\angle ABC = \angle BAD$. <i>Q. E. D.</i> | |

COROLLARY.

Hence, Every Equilateral Triangle is also Equiangular.

PER.

PROP VI.

If two Angles ABC, BAC , of a Triangle ABC , be equal, the one to the other, the Sides AC, BC , subtended under the equal Angles, shall also be equal.

PREPARATION.

If AC be said to be greater than BC , make (3.) $AE = BC$, if less, make (3.) $BD = AC$, and draw (post. 1.) AD, BE .

I say, $AC = BC$.

DEMONSTRATION.



- | | | |
|------------------------|----|--|
| For | 1 | If AC was $\sqsubset BC$. |
| Then | 2 | AB being common, $AE = BC$, $\angle BAC = \angle ABC$. <i>Const.</i> |
| $\therefore$ e. 2. | 3 | $\triangle ABE$ would be $= \triangle ABC$. 4. |
| But | 4 | $\triangle ABE \not\sqsubset \triangle ABC$. <i>Ax. 17.</i> |
| $\therefore$ | 5 | AC is not $\sqsubset BC$. |
| Again, | 6 | If AC was $\sqsupset BC$. |
| Then | 7 | AB being common, $BD = AC$, $\angle ABC = \angle BAC$. <i>Const.</i> |
| $\therefore$ e. 7. | 8 | $\triangle ABD$ would be $= \triangle ABC$. 4. |
| But | 9 | $\triangle ABD \not\sqsupset \triangle ABC$. <i>Ax. 17.</i> |
| $\therefore$ | 10 | AC is not $\sqsupset BC$. |
| $\therefore$ e. 5. 10. | 11 | AC is neither $\sqsubset$ nor $\sqsupset BC$. |
| $\therefore$ e. 11. | 12 | $AC = BC$. <i>Ax. 4. Q. E. D.</i> |

COROLLARY.

Hence, every Equiangular Triangle is also Equilateral.

Note. We shall here omit the Seventh Proposition, as only serving by Way of Lemma to the Eighth, which may be demonstrated without it.

PROP.

PROP. VIII.

If in two Triangles ABC, ABD, the two Sides AC, BC, of the one, be equal to the two Sides AD, BD, of the other, each to its correspondent Side (that is, $AC = AD$, and $BC = BD$) and the Base AB be common to them both, which is the same as to have equal Bases; then the Angles contained under the equal Sides shall be equal, viz. $\angle ACB = \angle ADB$, $\angle CAB = \angle DAB$, $\angle CBA = \angle DBA$.

PREPARATION.

From the Point C, to the Point D, in the Acute, Right, and Obtuse-Angled Triangles ABC, ABD, draw (post. 1.) the Right-line CD.

I say, $\angle ACB = \angle ADB$. $\angle CAB = \angle DAB$. $\angle CBA = \angle DBA$.



DEMONSTRATION.

- | | |
|---|--|
| 1 | $AC = AD$. Hyp. |
| 2 | $\angle ACD = \angle ADC$. 5. |
| 3 | $BC = BD$. Hyp. |
| 4 | $\angle BCD = \angle BDC$. 5. |
| 5 | $\angle ACD + \angle BCD = \angle ADC + \angle BDC$. Ax. 11. |
| 6 | $\angle ACB = \angle ADB$. |
| 7 | $AC = AD$, $BC = BD$, $\angle ACB = \angle ADB$. |
| 8 | $\angle CAB = \angle DAB$, $\angle CBA = \angle DBA$. 4. |
| 9 | $\angle ACB = \angle ADB$. $\angle CAB = \angle DAB$. $\angle CBA = \angle DBA$. Q. E. D. |

I

PROP.

PROP. IX.

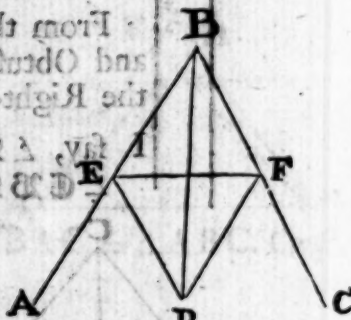
To divide an Angle ABC , into two equal Parts, or Angles ABD , CBD .

PREPARATION.

Take (3.) $BE = BF$, and draw (*post. 1.*) the Line EF ; upon which make (1.) an Equilateral Triangle EDF , and draw (*post. 4.*) the Right-line BD .

I say, $\angle ABD = \angle CBD$.

DEMONSTRATION.



- | | | |
|---|---|--------------------------|
| 1 | $BE = BF$, $DE = DF$, BD is common. | <i>Const.</i> |
| 2 | $\angle EBD = \angle FBD$. | 8. |
| 3 | $\angle ABD = \angle CBD$. | <i>Q. E. F. & D.</i> |

PROP. X.

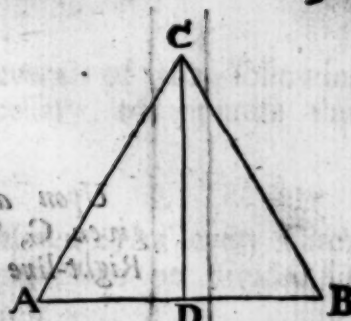
To Bisect a Right-line given AB .

PREPARATION.

Upon the Line given AB , erect (1.) an Equilateral Triangle ABC ; and (2.) Bisect the Angle C , with the Right-line CD .

I say, $AD = DB$

DEMONSTRATION.



- 1 $AC = BC$, DC is common, $\angle ACD = \angle BCD$. *Const.*
 2 $AD = DB$. 4 $Q.E.D.$

P R O P. XL

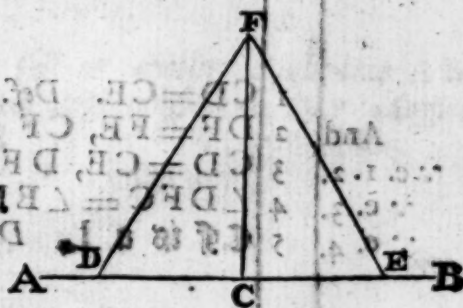
From a Point C , in a Right-line given AB , to erect a Perpendicular CE .

P R E P A R A T I O N.

Take (3.) on either Side of the Point given, $CD = CE$; then upon the Right-line DE , erect (1.) an Equilateral Triangle DEF , and draw (post. 1.) the Right-line CF .

I say, CF is a $\perp$

DEMONSTRATION.



- 1 $CD = CE$, $DF = EF$, CF is common. *Const.*
 2 $\angle DCF = \angle ECF$. 8.
 3 CF is a $\perp$. *Def. 10.* $Q.E.D.$

PROP.

P R O P. XII.

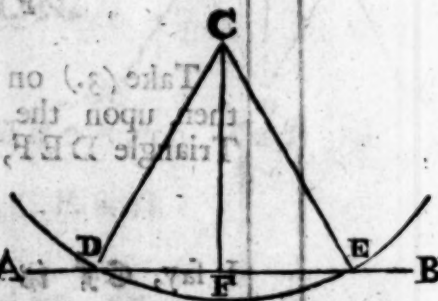
Upon an Infinite Right-line given AB, from a Point given C, that is not in it, to let fall a Perpendicular Right-line CF.

P R E P A R A T I O N.

From the given Point C, describe (*post. 3.*) a Circle cutting the Right-line given AB, in the Points D and E; then (*10.*) bisect DE in F, and draw (*post. 1.*) the Right-lines CD, CF, CE.

I say, CF is a $\perp$.

D E M O N S T R A T I O N.



1 $CD = CE$. *Def. 16.*

And 2 $DF = FE$, CF is common, *Const.*

3 $CD = CE$, $DF = FE$, CF is common.

4 $\angle DFC = \angle EFC$. 8.

5 CF is a $\perp$. *Def. 10. Q.E.F. & D.*

Note.

Note. In order to shorten several of the following Demonstrations, it will be necessary to premise thus much.

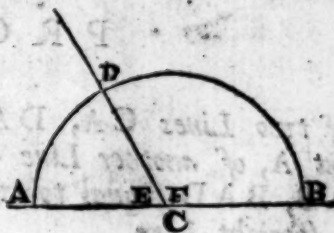
1st That the Periphery (or Circumference) of every Circle (whether Great or Small) is supposed to be divided into 360 equal Parts, called *Degrees*.

2^{dly} That all Angles are measured by the Arch of a Circle described upon the Angular Point, and are esteemed Greater or Less, according to the Number of Degrees contained in that Arch.

3^{dly} That a *Quadrant*, or *Quarter Part* of any Circle, is always $= 90^\circ$ Degrees, being the Measure of a Right-angle; and that a Semi-circle is $= 180^\circ$ Degrees, being the Measure of two Right-angles; also that equal Arches of a Circle, or of equal Circles, measure equal Angles.

P R O P. XIII.

If one Right-line DC, fall on another Right-line AB, it will make with it two Right-angles, or two Angles equal to two Right-angles.



K

P R E

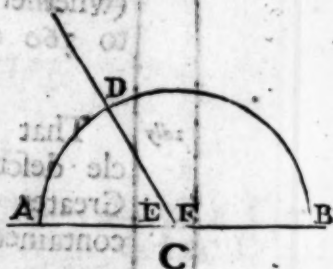
PREPARATION.

If the Angles E, F, be equal, then (*Def.* 10.) they make two Right-angles; if unequal, then from the Point C, describe any Semicircle.

I say, $\angle E + \angle F = 2r\angle$.

DEMONSTRATION.

- 1 Arch AD = $\angle E$.
 And 2 Arch DB = $\angle F$.
 1 + 2 3 Arch AD + arch DB = $\angle E + \angle F$. *Ax.* 11.
 Also 4 Arch AD + arch DB = $1r\angle$.
 $\therefore$ e. 3. 4 5 $\angle E + \angle F = 2r\angle$. *Ax.* 1. Q. E. D.



COROLLARIES.

- 1 Hence, If $\angle F = 90^\circ$, then $\angle E = 90^\circ$. but if $\angle F$ be Obtuse, or $\square 90^\circ$, then $\angle E$ will be Acute, or $\square 90^\circ$.
 2 If several Right-lines stand upon, or meet with any Right-line at one and the same Point, as C, all the Angles taken together will be $= 2r\angle$; *viz.* 180° .

PROP. XIV.

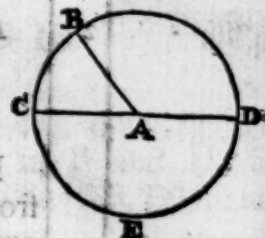
If two Lines CA, DA, meeting together at the same Point A, of another Line AB, make with it two Angles CAB, BAD, equal to two Right ones; they will make one straight Line.

P R E

PREPARATION.

From the Center A, describe a Circle CBDE.

I say, CAD is a straight Line.



DEMONSTRATION.

- 1 $\angle CAB + \angle BAD = 2r\angle$. Hyp.
- 2 Arch CBD is a Semicircle.
- 3 CAD is the Diameter.
- 4 CAD is a straight Line. Def. 17. Q.E.D.

PROP. XV.

If two Lines AB, DE, intersect or cross each other, the two opposite Angles at the Vertex will be equal.

That is, $\angle B = \angle F$. $\angle D = \angle C$.

DEMONSTRATION.



- 1 $\angle G + \angle C = 2r\angle$. 13.
- 2 $\angle F + \angle C = 2r\angle$. 13.
- 3 $\angle G + \angle C = \angle F + \angle C$. Ax. 1.
- 4 $\angle G = \angle F$. Ax. 12.
- 5 $\angle K + \angle F = 2r\angle$. 13.
- 6 $\angle K + \angle F = \angle F + \angle C$. Ax. 1.
- 7 $\angle K = \angle C$. Ax. 12.
- 8 $\angle B = \angle F$. $\angle D = \angle C$. Q.E.D.

K 3

COROL.

COROLLARIES.

Hence, Two Right-lines crossing each other make Angles equal to four Right.

All the Angles made about one Point make four Right.

SCHOLIUM.

If four Right-lines EA, EB, EC, ED, proceeding from one Point E, make the Angles vertically opposite equal the one to the other, each two Lines, AE, EB, and CE, ED, are placed in one straight Line.

That is, AEB, CED, are straight Lines.

DEMONSTRATION.



- | | | |
|---------------------|--|--------------------|
| 1 | $\angle AEC = \angle BED.$ | <i>Hyp.</i> |
| And 2 | $\angle CEB = \angle DEA.$ | <i>Hyp.</i> |
| 1 + 2 3 | $\angle AEC + \angle CEB = \angle BED + \angle DEA.$ | <i>Ax. 11.</i> |
| Again 4 | $\angle AEC + \angle CEB + \angle BED + \angle DEA = 4r \angle.$ | <i>Cor. 2. 15.</i> |
| 4 - 3 5 | $\angle BED + \angle DEA = 4r \angle - \angle AEC - \angle CEB.$ | <i>Ax. 12.</i> |
| 5 + $\angle BED$ 6 | $2\angle BED + \angle DEA = 4r \angle - \angle DEA.$ | <i>Ax. 11.</i> |
| 6 + $\angle DEA$ 7 | $2\angle BED + 2\angle DEA = 4r \angle.$ | <i>Ax. 11.</i> |
| 7 ÷ 2 8 | $\angle BED + \angle DEA = 2r \angle.$ | <i>Ax. 14.</i> |
| ∴ e. 8. 9 | ABB is a straight Line. | 14. |
| Lastly, e. 8. 2. 10 | $\angle BED + \angle CEB = 2r \angle.$ | <i>Sub.</i> |
| ∴ e. 10. 11 | CED is a straight Line. | 14. |
| ∴ e. 9. 11. 12 | AEB, CED, are straight Lines. | Q.E.D. |

PROP.

PROP. XVI.

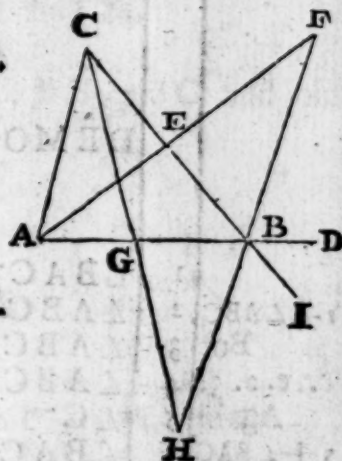
AB, one of the Sides of a Triangle ABC, being produced, the exterior Angle CBD, is greater than either of the interior or opposite Angles ACB, CAB.

PREPARATION.

Make (ro.) $CE = EB$, $AG = GB$, and drawing the Lines AE, CG, and extending them to F and H, so that $AE = EF$, and $CG = GH$, join BF, BH, and extend CB to I.

I say, $\angle CBD > \angle ACB$ or $\angle CAB$.

DEMONSTRATION.



- | | | | |
|-------------------------|----|--|---------|
| | 1 | $AE = EF, CE = EB.$ | Const. |
| And | 2 | $\angle AEC = \angle BEF.$ | 15. |
| $\therefore$ e. 1. 2. | 3 | $AE = EF, CE = EB, \angle AEC = \angle BEF.$ | |
| $\therefore$ e. 3. | 4 | $\angle ACE = \angle EBF.$ | 4. |
| But | 5 | $\angle CBD > \angle EBF.$ | Ax. 17. |
| $\therefore$ e. 4. 5. | 6 | $\angle CBD > \angle ACE.$ | Ax. 2. |
| i.e. | 7 | $\angle CBD > \angle ACB.$ | |
| Again | 8 | $CG = GH, AG = GB.$ | Const. |
| And | 9 | $\angle AGC = \angle BGH.$ | 15. |
| $\therefore$ e. 8. 9. | 10 | $CG = GH, AG = GB, \angle AGC = \angle BGH.$ | |
| $\therefore$ e. 10. | 11 | $\angle CAG = \angle GBH.$ | 4. |
| But | 12 | $\angle GBI > \angle GBH.$ | Ax. 17. |
| $\therefore$ e. 11. 12. | 13 | $\angle GBI > \angle CAG.$ | Ax. 2. |
| Now. | 14 | $\angle GBI = \angle CBD.$ | 15. |
| $\therefore$ e. 13. 14. | 15 | $\angle CBD > \angle CAG.$ | Ax. 3. |
| i.e. | 16 | $\angle CBD > \angle CAB.$ | |
| $\therefore$ e. 7. 16. | 17 | $\angle CBD > \angle ACB$ or $\angle CAB.$ | Q.E.D. |

L

PROP.

PROP. XVII.

Two Angles of any Triangle ABC, which way soever they are taken, are less than two Right-angles.

PREPARATION.

Produce the Side AB both ways.

I say, $\angle ABC + \angle BAC$ or $\angle BAC + \angle C$ or $\angle C + \angle ABC = 2r\angle$.

DEMONSTRATION



- | | | | | |
|---------------------------|----|--|--------|---------|
| | 1 | $\angle BAC = \angle CBD$. | 16. | |
| 1 + $\angle ABC$ | 2 | $\angle ABC + \angle BAC = \angle ABC + \angle CBD$. | | Ax. 15. |
| But | 3 | $\angle ABC + \angle CBD = 2r\angle$. | 13. | |
| $\therefore$ e. 2. 3. | 4 | $\angle ABC + \angle BAC = 2r\angle$. | Ax. 2. | |
| Again | 5 | $\angle C = \angle CAE$. | 16. | |
| 3 + $\angle BAC$ | 6 | $\angle BAC + \angle C = \angle BAC + \angle CAE$. | | Ax. 15. |
| But | 7 | $\angle BAC + \angle CAE = 2r\angle$. | 13. | |
| $\therefore$ e. 6. 7. | 8 | $\angle BAC + \angle C = 2r\angle$. | Ax. 2. | |
| Lastly | 9 | $\angle C = \angle CBD$. | 16. | |
| 9 + $\angle ABC$ | 10 | $\angle C + \angle ABC = \angle CBD + \angle ABC$. | | Ax. 15. |
| But | 11 | $\angle CBD + \angle ABC = 2r\angle$. | 13. | |
| $\therefore$ e. 10. 11. | 12 | $\angle C + \angle ABC = 2r\angle$. | Ax. 2. | |
| $\therefore$ e. 4. 8. 12. | 13 | $\angle ABC + \angle BAC$ or $\angle BAC + \angle C$ or $\angle C + \angle ABC = 2r\angle$. | | Q.E.D. |

COROL.

COROLLARIES.

1 Hence, In every Triangle, wherein One Angle is either Right or Obtuse, the two others are Acute Angles.

2 All the Angles of an Equilateral Triangle, and the two Angles of an Isosceles Triangle that are upon the Base, are Acute.

PROP. XVIII.

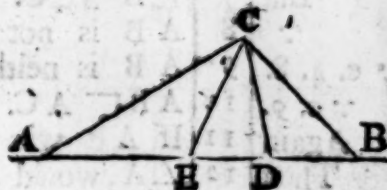
The greatest Side AB, of every Triangle ABC, subtends the greatest Angle ACB.

PREPARATION.

From AB, take (3.) $AD = AC$, $BE = BC$, and join CD, CE.

I say, $\angle ACB \square \angle B$ or $\angle A$.

DEMONSTRATION.



	1	$AD = AC$, Const.	
$\therefore$ e. 1.	2	$\angle ADC = \angle ACD$. 5.	
Also	3	$\angle ADC \square \angle B$. 16.	
$\therefore$ e. 2. 3.	4	$\angle ACD \square \angle B$. Ax. 3.	
But	5	$\angle ACB \square \angle ACD$. Ax. 17.	
$\therefore$ e. 4. 5.	6	$\angle ACB$ is much $\square \angle B$.	
Again	7	$BE = BC$. Const.	
$\therefore$ e. 7.	8	$\angle BEC = \angle BCE$. 5.	
Also	9	$\angle BEC \square \angle A$. 16.	
$\therefore$ e. 8. 9.	10	$\angle BCE \square \angle A$. Ax. 3.	
But	11	$\angle ACB \square \angle BCE$. Ax. 17.	
$\therefore$ e. 10. 11.	12	$\angle ACB$ is much $\square \angle A$.	
$\therefore$ e. 6. 12.	13	$\angle ACB \square \angle B$ or $\angle A$. Q. E. D.	

L 2

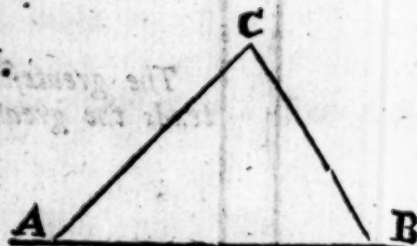
PROP.

PROP. XIX.

In every Triangle ABC , under the greatest Angle C ,
is subtended the greatest Side AB .

That is, $AB \sqsupset AC$ or BC .

DEMONSTRATION.



- | | | |
|-------------------------|----|--|
| For | 1 | If AB was $= AC$. |
| Then | 2 | $\angle B$ would be $= \angle C$. 5. |
| But | 3 | $\angle B \sqsupset \angle C$. Hyp. |
| $\therefore$ | 4 | AB is not $= AC$. |
| Also | 5 | If AB was $\sqsupset AC$. |
| Then | 6 | $\angle B$ would be $\sqsupset \angle C$. 18. |
| But | 7 | $\angle B \sqsupset \angle C$. Hyp. |
| $\therefore$ | 8 | AB is not $\sqsupset AC$. |
| $\therefore$ e. 4. 8. | 9 | AB is neither $=$ nor $\sqsupset AC$. |
| $\therefore$ e. 9. | 10 | $AB \sqsupset AC$. Ax. 5. |
| Again | 11 | If AB was $= BC$. |
| Then | 12 | $\angle A$ would be $= \angle C$. 5. |
| But | 13 | $\angle A \sqsupset \angle C$. Hyp. |
| $\therefore$ | 14 | AB is not $= BC$. |
| Also | 15 | If AB was $\sqsupset BC$. |
| Then | 16 | $\angle A$ would be $\sqsupset \angle C$. 18. |
| But | 17 | $\angle A \sqsupset \angle C$. Hyp. |
| $\therefore$ | 18 | AB is not $\sqsupset BC$. |
| $\therefore$ e. 14. 18. | 19 | AB is neither $=$ nor $\sqsupset BC$. |
| $\therefore$ e. 19. | 20 | $AB \sqsupset BC$. Ax. 5. |
| $\therefore$ e. 10. 20. | 21 | $AB \sqsupset AC$ or BC . Q. E. D. |

PROP.

PROP. XX.

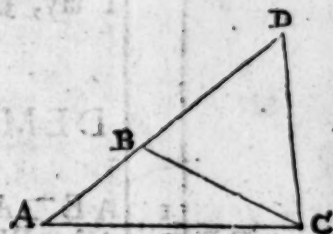
Of any Triangle ABC , two Sides any way taken, are greater than that which remains; so that the Sum of the lesser Sides AB , BC , is greater than the greatest AC .

PREPARATION.

Produce AB ; and taking (3.) $BD = BC$, join DC .

I say, $AB + BC > AC$.

DEMONSTRATION.



- | | | |
|-----------------------|---|--|
| | 1 | $BD = BC$. <i>Const.</i> |
| $\therefore$ e. 1. | 2 | $\angle D = \angle BCD$. 5. |
| But | 3 | $\angle ACD < \angle BCD$. <i>Ax. 17.</i> |
| $\therefore$ e. 2. 3. | 4 | $\angle ACD < \angle D$. <i>Ax. 2.</i> |
| $\therefore$ e. 4. | 5 | $AD < AC$. 19. |
| i. e. | 6 | $AB + BD < AC$. |
| i. e. 1. 6. | 7 | $AB + BC < AC$. <i>Sub. Q. E. D.</i> |

M

PROP.

P R O P. XXI.

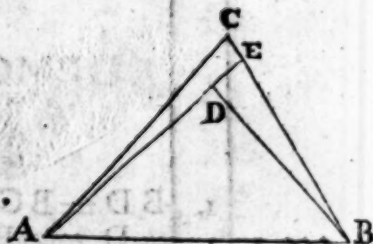
If from the Point D, taken at Pleasure in any Triangle ABC, the Right-lines DB, DA, be drawn to the Extrems B, A, of one of its Sides AB, their Sum DB + DA, will be less than the Sum AC + CB, of the other two Sides AC, CB; but the Angle ADB, will be greater than the Angle ACB.

P R E P A R A T I O N.

Extend DA to E.

I say, $DB + DA < AC + CB$. $\angle ADB > \angle ACB$.

D E M O N S T R A T I O N.



- | | | |
|-------------|----|--|
| | 1 | $AE < AC + CE$. 20. |
| 1 + EB. | 2 | $AE + EB < AC + CE + EB$. Ax. 15. |
| i.e. | 3 | $AE + EB < AC + CB$. |
| Again | 4 | $DB < EB + ED$. 20. |
| 4 + DA. | 5 | $DB + DA < EB + ED + DA$. Ax. 15. |
| i.e. | 6 | $DB + DA < AE + EB$. |
| But e. 3. | 7 | $AE + EB < AC + CB$. |
| ∴ e. 6. 7. | 8 | $DB + DA$ is much $< AC + CB$. |
| Lastly | 9 | $\angle ADB < \angle DEB$. 16. |
| Also | 10 | $\angle DEB < \angle ACE$. 16. |
| ∴ e. 9. 10. | 11 | $\angle ADB$ is much $< \angle ACE$. |
| i.e. | 12 | $\angle ADB$ is much $< \angle ACB$. |
| ∴ e. 8. 12. | 13 | $DB + DA < AC + CB$. $\angle ADB > \angle ACB$. Q. E. D. |

PROP.

P R O P. XXII.

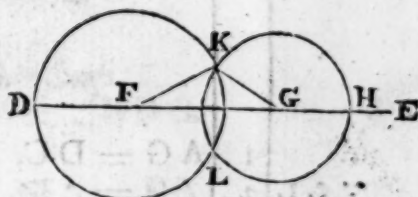
Of three given Right-lines A, B, C, to make a Triangle KFG, whereof the biggest ought (20.) to be less than the Sum of the other two.

P R E P A R A T I O N.

Draw the Infinite Right-line DE, and from it take (3.) $FD = A$, $FG = B$, $GH = C$; then from the Centers F and G, with the Radij FD, GH, describe (post. 3.) the Circles DKL, HKL, cutting each other in the Point K, and from that Point draw KF, KG.

I say, $\triangle KFG$ is the $\triangle$ sought.

D E M O N S T R A T I O N.



- | | |
|----------|--|
| 1 | $FD = KF$. Def. 16. |
| Alfo 2 | $FD = A$. Const. |
| 3 | $KF = A$. Ax. 1. A ——— B ——— C ——— |
| Again 4 | $FG = B$. Const. |
| Lastly 5 | $GH = KG$. Def. 16. |
| Alfo 6 | $GH = C$. Const. |
| 7 | $KG = C$. Ax. 1. |
| 8 | $KF = A$, $FG = B$, $KG = C$. |
| 9 | $\triangle KFG$ is the $\triangle$ sought. Q. E. F. & D. |

PROP. XXIII.

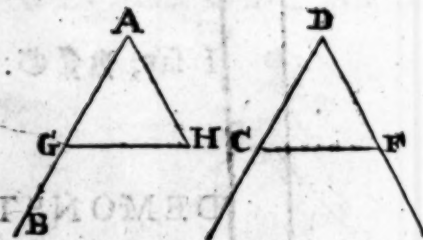
At a Point A , in a Right-line given AB , to make a Right-lined Angle A , equal to a Right-lined Angle given D .

PREPARATION.

Draw the Right-line CF , cutting the Sides of the Angle given any ways; make (3.) $AG = DC$; upon A G , raise (22.) a Triangle AGH , Equilateral to the former DCF , so that $AH = DF$, and $GH = CF$.

I say, $\angle A = \angle D$.

DEMONSTRATION.



$\therefore$ c. 1. 2 $AG = DC$, $AH = DF$, $GH = CF$. Const.
 $\angle A = \angle D$. 8. Q. E. F. & D.

PROP. XXIV.

If two Triangles ABC , DEF , have two Sides of the one Triangle AB , AC , equal to two Sides of the other Triangle DE , DF , each to its correspondent Side, (that is, $AB = DE$, and $AC = DF$), and have the Angle A greater than the Angle E , comprehended under the equal Right-lines, they shall also have the Base BC greater than the Base EF .

PRE-

PREPARATION.

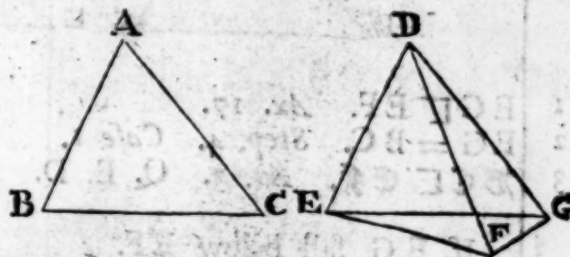
Make (23.) $\angle EDG = \angle A$, and (3.) $DG = DF =$
(Hyp.) AC , and join EG, FG .

Case. I

If EG fall above EF .

I say, $\angle B < \angle C$.

DEMONSTRATION.



- | | | |
|-------------------------|----|---|
| | 1 | $AB = DE$. Hyp. |
| And | 2 | $AC = DG, \angle A = \angle EDG$. Const. |
| $\therefore$ e. 1. 2. | 3 | $AB = DE, AC = DG, \angle A = \angle EDG$. |
| $\therefore$ e. 3. | 4 | $BC = EG$. 4. |
| Again | 5 | $DG = DF$. Const. |
| $\therefore$ e. 5. | 6 | $\angle DGF = \angle DFG$. 5. |
| But | 7 | $\angle DGF < \angle EGF$. Ax. 17. |
| $\therefore$ e. 6. 7. | 8 | $\angle DFG < \angle EGF$. Ax. 3. |
| Now | 9 | $\angle EFG < \angle DFG$. Ax. 17. |
| $\therefore$ e. 8. 9. | 10 | $\angle EFG$ is much $<$ $\angle EGF$. |
| $\therefore$ e. 10. | 11 | $EG < EF$. 19. |
| But e. 4. | 12 | $EG = BC$. |
| $\therefore$ e. 11. 12. | 13 | $\angle B < \angle C$. Ax. 3. Q.E.D. |

Case. 2 If EG fall upon EF.

I say, $\angle BCE = \angle GEF$.

DEMONSTRATION.

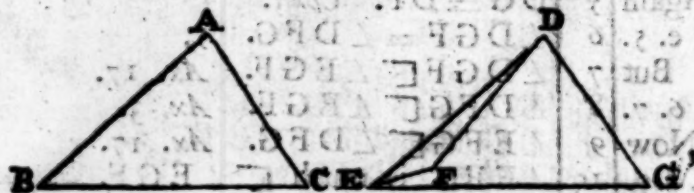


- 1 $EG \sqsubset EF$. Ax. 17.
 But 2 $EG = BC$. Step. 4. Case 1.
 ∴ c. 1. 2. 3 $\angle BCE = \angle GEF$. Ax. 3. Q. E. D.

Case. 3 If EG fall below EF.

I say $\angle BCE = \angle GEF$.

DEMONSTRATION.



- 1 $DG + EG \sqsubset DF + EF$. 21.
 And 2 $DG = DF$. Const.
 ∴ c. 1. 2. 3 $DG + EG \sqsubset DG + EF$. Sub.
 3 - DG 4 $EG \sqsubset EF$. Ax. 16.
 But 5 $EG = BC$. Step 4. Case 1.
 ∴ c. 4. 5. 6 $\angle BCE = \angle GEF$. Ax. 3. Q. E. D.

PROP.

PROP. XXV.

If two Triangles ABC , DEF , have two Sides AB , AC , equal to two Sides DE , DF , each to its correspondent Side, (that is, $AB = DE$, and $AC = DF$) and have the Base BC greater than the Base EF , they shall also have the Angle A , comprehended under the equal Right-lines, greater than the Angle D .

That is, $\angle A \square \angle D$

DEMONSTRATION.



- | | | | |
|----------------------|----|---|------------------------|
| For | 1 | If $\angle A$ was $= \angle D$. | |
| Then | 2 | AB being $= DE$, $AC = DF$, $\angle A = \angle D$. | <i>Hyp.</i> |
| $\therefore$ e. 2. | 3 | BC would be $= EF$. | 4. |
| But | 4 | $BC \square EF$. | <i>Hyp.</i> |
| | 5 | $\angle A$ is not $= \angle D$. | |
| Again | 6 | If $\angle A$ was $\square \angle D$. | |
| Then | 7 | BC would be $\square EF$. | 24. |
| But | 8 | $BC \square EF$. | <i>Hyp.</i> |
| | 9 | $\angle A$ is not $\square \angle D$. | |
| $\therefore$ e. 5. 9 | 10 | $\angle A$ is neither $=$ nor $\square \angle D$. | |
| $\therefore$ e. 10. | 11 | $\angle A \square \angle D$. | <i>Ax. 5. Q. E. D.</i> |

P R O P. XXVI.

If two Triangles ABC , DEF , have two Angles of the one, A , ABC , equal to two Angles of the other D , DEF , each to its correspondent Angle, (that is, $\angle A = \angle D$, and $\angle ABC = \angle DEF$) and have also one Side of the one, equal to one Side of the other, either that Side which is comprehended betwixt the equal Angles, or that which is subtended under one of the equal Angles; the other Sides also of the one, shall be equal to the other Sides of the other, each to its correspondent Side, and the other Angle of the one, shall be equal to the other Angle of the other, and the two Triangles entirely equal.

Hyp. I

Suppose $AB = DE$, which are the Sides comprehend-
ed betwixt the equal Angles.

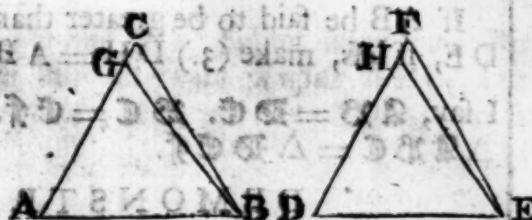
P R E P A R A T I O N.

If AC be said to be greater than DF , make (3.) $AG = DF$, if less, make (3.) $DH = AC$, and draw BG , EH .

I say, $AC = DF$. $BC = EF$. $\angle ACB = \angle DFE$,
 $\triangle ABC = \triangle DEF$.

DEMON-

DEMONSTRATION.



- For 1 If AC was $\sqsubset$ DF.
 Then 2 AG being = DF, AB = DE, $\angle A = \angle D$. Hyp.
 ∴ e. 2 3 $\angle ABG$ would be = $\angle DEF$. 4.
 Also 4 $\angle ABC = \angle DEF$. Hyp.
 ∴ e. 3. 4. 5 $\angle ABG$ would be = $\angle ABC$. Ax. 1.
 But 6 $\angle ABG \sqsubset \angle ABC$. Ax. 17.
 ∴ 7 AC is not $\sqsubset$ DF.
 Again 8 If AC was $\sqsupset$ DF.
 Then 9 AC being = DH, AB = DE, $\angle A = \angle D$. Hyp.
 ∴ e. 9. 10 $\angle ABC$ would be = $\angle DEH$. 4.
 Also 11 $\angle ABC = \angle DEF$. Hyp.
 ∴ e. 10. 11. 12 $\angle DEH$ would be = $\angle DEF$. Ax. 1.
 But 13 $\angle DEH \sqsupset \angle DEF$. Ax. 17.
 ∴ 14 AC is not $\sqsupset$ DF.
 ∴ e. 7. 14. 15 AC is neither $\sqsubset$ nor $\sqsupset$ DF.
 ∴ e. 15. 16 AC = DF. Ax. 4.
 ∴ e. 16. 17 AC = DF, AB = DE, $\angle A = \angle D$.
 ∴ e. 17. 18 AC = DF, BC = EF, $\angle ACB = \angle DEF$. ∴ e. $\triangle ABC = \triangle DEF$. 4. Q. E. D.

Suppose

Hyp. 2

Suppose $AC = DF$, which are the Sides subtended under the equal Angles.

PREPARATION.

If AB be said to be greater than DE , make (3.) $AG = DE$, if less, make (3.) $DH = AB$, and draw CG, FH .

I say, $AB = DE$. $BC = EF$. $\angle ACB = \angle DFE$.
 $\triangle ABC = \triangle DEF$.

DEMONSTRATION.



- For 1 If AB was $\sqsubset DE$.
 Then 2 AG being $= DE$, $AC = DF$, $\angle A = \angle D$. Hyp.
 $\therefore$ e. 2. 3 $\angle AGC$ would be $= \angle DEF$. 4.
 Also 4 $\angle ABC = \angle DEF$. Hyp.
 $\therefore$ e. 3. 4. 5 $\angle AGC$ would be $= \angle ABC$. Ax. 1.
 But 6 $\angle AGC \sqsubset \angle ABC$. 16.
 $\therefore$ 7 AB is not $\sqsubset DE$.
 Again 8 If AB was $\sqsupset DE$.
 Then 9 AB being $= DH$, $AC = DF$, $\angle A = \angle D$. Hyp.
 $\therefore$ e. 9. 10 $\angle ABC$ would be $= \angle DHF$. 4.
 Also 11 $\angle ABC = \angle DEF$. Hyp.
 $\therefore$ e. 10. 11. 12 $\angle DHF$ would be $= \angle DEF$. Ax. 1.
 But 13 $\angle DHF \sqsubset \angle DEF$. 16.
 $\therefore$ 14 AB is not $\sqsupset DE$.
 $\therefore$ e. 7. 14. 15 AB is neither $\sqsubset$ nor $\sqsupset DE$.
 $\therefore$ e. 15. 16 $AB = DE$. Ax. 4.
 $\therefore$ e. 16. 2. 17 $AB = DE$, $AC = DF$, $\angle A = \angle D$.
 $\therefore$ e. 17. 18 $AB = DE$. $BC = EF$. $\angle ACB = \angle DFE$.
 $\triangle ABC = \triangle DEF$. 4 Q. E. D.

PROP.

PROP. XXVII.

If a Right-line EF, falling upon two Right-lines AB, CD, make the alternate Angles AEF, DFE, equal the one to the other, then are the Right-lines AB, CD, parallel.

PREPARATION.

If AB, CD, be said not to be parallel, suppose them to be produced 'till they meet in some Point G, so will EGF be a Triangle.

I say, $AB \parallel CD$.

DEMONSTRATION.



- For 1 If AB, CD, being produced, should ever meet; as in G.
 Then 2 $\angle AEF$ would be $\sphericalangle \angle DFE$. 16.
 But 3 $\angle AEF = \angle DFE$. Hyp.
 $\therefore$ 4 AB, CD, being produced, will never meet; as in G.
 $\therefore$ 5 $AB \parallel CD$. Def. 34. Q. E. D.

PROPOSITION XXVIII.

If one Right line GF , cut two other Right-lines AB , CD , so that the exterior Angle BEG , be equal to the interior and opposite on the same Side DFE , or that the two interior on the same Side BEF , DFE , be together equal to two right Ones, those two Right-lines will be parallel.

Hyp. Suppose $\angle BEG = \angle DFE$.

DEMONSTRATION.



- Also
- 1 $\angle BEG = \angle AEF$. 15.
 - 2 $\angle BEG = \angle DFE$. Hyp.
 - 3 $\angle AEF = \angle DFE$. Ax. 1.
 - 4 $AB \parallel CD$. 27. Q. E. D.

Hyp. Suppose $\angle BEF + \angle DFE = 2r\angle$.

I say, $AB \parallel CD$.

DEMONSTRATION.

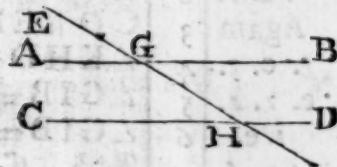
- Also
- 1 $\angle BEF + \angle AEF = 2r\angle$. 13.
 - 2 $\angle BEF + \angle DFE = 2r\angle$. Hyp.
 - 3 $\angle BEF + \angle AEF = \angle BEF + \angle DFE$. Ax. 1.
 - 4 $\angle AEF = \angle DFE$. Ax. 12.
 - 5 $AB \parallel CD$. 27. Q. E. D.

PROP.

XXX. P R O P. XXIX.

If a Right-line EH, intersect two Parallels AB, CD, the alternate Angles AGH, GHD, will be equal to one another; the exterior Angle EGB, will be equal to the interior and opposite on the same Side GHD, and the two interiors on the same Side BGH, GHD, will be together equal to two Right-angles.

That is, $\angle AGH = \angle GHD$. $\angle EGB = \angle GHD$.
 $\angle BGH + \angle GHD = 2r\angle$.



DEMONSTRATION.

- | | |
|---|---|
| 1 | $AB \parallel CD$. Hyp. |
| 2 | $\angle BGH + \angle GHD = 2r\angle$. Ax. 22. |
| 3 | $\angle BGH + \angle AGH = 2r\angle$. 13. |
| 4 | $\angle BGH + \angle AGH = \angle BGH + \angle GHD$. Ax. 1. |
| 5 | $\angle AGH = \angle GHD$. Ax. 12. |
| 6 | $\angle AGH = \angle EGB$. 15. |
| 7 | $\angle EGB = \angle GHD$. Ax. 1. |
| 8 | $\angle AGH = \angle GHD$, $\angle EGB = \angle GHD$, $\angle BGH + \angle GHD = 2r\angle$. Q. E. D. |

P

PROP.

PROP. XXX.

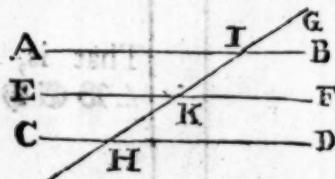
Right-lines AB, CD, which are parallel to one and the same Right-line EF, are parallel to each other.

PREPARATION.

Draw GH cutting the three Right-lines any ways.

I say, $AB \parallel CD$.

DEMONSTRATION.



- $\therefore$ e. 1. 1 $AB \parallel EF$. Hyp.
 Again 2 $\angle GIB = \angle IKF$. 29.
 $\therefore$ e. 3. 3 $CD \parallel EF$. Hyp.
 $\therefore$ e. 2. 4. 4 $\angle KHD = \angle IKF$. 29.
 i.e. 5 $\angle GIB = \angle KHD$. Ax. 1.
 $\therefore$ e. 6. 6 $\angle GIB = \angle IHD$.
 7 $AB \parallel CD$. 28. Q. E. D.

PROP. XXXI.

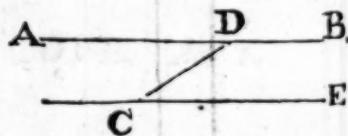
To draw thro' a given Point C, a Right-line parallel to a given Right-line AB.

PREPARATION.

From the given Point C, draw a Right-line CD; to any Point of the given Right-line AB; then make (23.) at the Point C, the Angle DCE equal to the Angle ADC.

I say, $CE \parallel AB$.

DEMONSTRATION.



- $\therefore$ e. 1. 1 $\angle DCE = \angle ADC$. Const.
 2 $CE \parallel AB$. 27. Q. E. F. & D.

PROP.

PROP. XXXII.

In any Triangle ABC, the $\angle A + \angle B + \angle C$, viz. the three Angles taken together, are equal to two Right-angles; and AC one of the Sides, being produced, the exterior $\angle Z = \angle A + \angle B$, viz. the two interior and opposite Ones taken together.

PREPARATION.

Draw (31.) HK || AD, just touching the Vertical Angle B; then produce the Sides AB, CB.

I say, $\angle A + \angle B + \angle C = 2r\angle^s$. $\angle Z = \angle A + \angle B$.

DEMONSTRATION.



- | | | |
|--------------------|--|-------------|
| 1 | $\angle E = \angle A$. | 29. |
| And 2 | $\angle F = \angle B$. | 15. |
| And 3 | $\angle G = \angle C$. | 29. |
| 1+2+3 4 | $\angle E + \angle F + \angle G = \angle A + \angle B + \angle C$. | Ax. 11. |
| Also 5 | $\angle E + \angle F + \angle G = 2r\angle^s$. | Cor. 2. 13. |
| ∴ e. 4. 5. 6 | $\angle A + \angle B + \angle C = 2r\angle^s$. | Ax. 1. |
| Again. 7 | $\angle C + \angle Z = 2r\angle^s$. | 13. |
| ∴ e. 7. 6. 8 | $\angle C + \angle Z = \angle A + \angle B + \angle C$. | Ax. 1. |
| 8 - $\angle C$. 9 | $\angle Z = \angle A + \angle B$. | Ax. 12. |
| ∴ e. 6. 9. 10 | $\angle A + \angle B + \angle C = 2r\angle^s$. $\angle Z = \angle A + \angle B$. | |

Q. E. D.

COROLLARIES.

- 1 Because the $3\angle$ ' of every Δ are $= 2r\angle$, therefore the $3\angle$ ' of one Δ , are $=$ the $3\angle$ ' of another Δ .
- 2 If $2\angle$ ' of one Δ , are $= 2\angle$ ' of another Δ , each to each, the third $\angle$ of the one, will be $=$ the third $\angle$ of the other.
- 3 In a Right-angled Δ , the two Acute $\angle$ ' taken together, are precisely $= a r\angle$, viz. 90° Degrees.
Consequently, if one of the Acute $\angle$ ' be given, the other is also given; viz. 90° Degrees — the given $\angle$ leaves the other $\angle$.
- 4 Each $\angle$ of an Equilateral Δ is $= 60^\circ$ Degrees, or the third of $2r\angle$, which make 180° Degrees.
- 5 If one $\angle$ of any plain Δ be given, the Sum of the other $2\angle$ ' is also given, since 180° Degrees — the given $\angle$ is $=$ the other $2\angle$ '.
- 6 In an Isosceles Δ , if the $\angle$ comprehended between the equal Legs be $= a r\angle$, each of those at the Base will be $= \frac{1}{2}r\angle$.

THE-

THEOREM I.

All the Angles of a Right-lined Figure do together make twice as many Right-angles, bating four, as there are Sides of the Figure.

PREPARATION.

From any Point within the Figure, let Right-lines be drawn to all the Angles of the Figure, which shall resolve the Figure into as many Triangles as there are Sides of the Figure.

I say, All the $\angle^s$ of the Figure are = twice as many $r\angle^s$, bating four, as there are Sides.

DEMONSTRATION.



- 1 The $\angle^s$ of every $\triangle$ in the Figure are = $2r\angle^s$. 32.
- 2 The $\angle^s$ of all the $\triangle^s$ in the Figure are = twice as many $r\angle^s$ as there are Sides.
- But 3 The $\angle^s$ at the Point within the Figure are = $4r\angle^s$. Cor. 2. 15.
- 4 The remaining $\angle^s$ are = twice as many $r\angle^s$, bating four, as there are Sides.
- i.e. 5 All the $\angle^s$ of the Figure are = twice as many $r\angle^s$, bating four, as there are Sides. Q. E. D.

Q

COROL.

COROLLARIES.

1 Hence, all Right-lined Figures of the same Species have the Sums of their Angles equal.

2 In a Figure of four Sides, the four Angles make together four Right.

THEOR. II.

All the outward Angles of any Right-lined Figure, taken together, make up four Right-angles.

PREPARATION.

Produce the Sides of the Figure.

I say, The outward $\angle^s$ are $= 4r\angle^s$.

DEMONSTRATION.



1 All the several Inward, with the several Outward $\angle^s$ of the Figure are $= 2r\angle^s$. 13.

2 All the Inward, together with the Outward $\angle^s$ of the Figure, are $=$ twice as many $r\angle^s$ as there are Sides.

But 3 All the Inward $\angle^s$, with four right, are $=$ twice as many $r\angle^s$ as there are Sides. Theor. 1.

4 The outward $\angle^s$ are $= 4r\angle^s$. Q. E. D.

COROL.

COROLLARY.

All Right-lined Figures of whatsoever Species have the Sums of their outward Angles equal.

PROP. XXXIII.

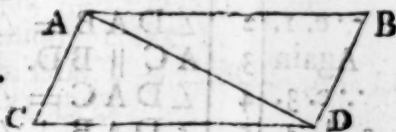
If two Lines AB , CD , are parallel and equal, the Right-lines AC , BD , which join their Extremities, are also parallel and equal.

PREPARATION.

Draw the Diagonal AD .

I say, $AC \parallel$ and $= BD$.

DEMONSTRATION.



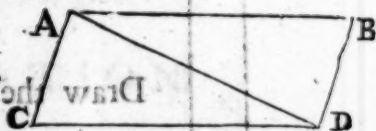
- | | |
|-----------------|---|
| 1 | $AB = CD$. Hyp. |
| And 2 | AD is common. |
| And 3 | $\angle ADC = \angle DAB$ 29. |
| ∴ c. 1. 2. 3. 4 | $AB = CD$, AD is common, $\angle ADC = \angle DAB$. |
| ∴ c. 4. 5 | $AC = BD$. 4. |
| Also c. 4. 6 | $\angle DAC = \angle ADB$. 4. |
| ∴ c. 6. 7 | $AC \parallel BD$. 27. |
| ∴ c. 7. 5. 8 | $AC \parallel$ and $= BD$. Q. E. D. |

PROP. XXXIV.

In Parallelograms, as $ABDC$, the opposite Sides AB , CD , and AC , BD , are equal each to other; and the opposite Angles, C , B , and CAB , CDB , are also equal; and the Diagonal AD , divides the same into two equal Parts or Triangles ADB , ADC .

That is, $AB = CD$, $AC = BD$, $\angle C = \angle B$, $\angle CAB = \angle CDB$. $\triangle ADB = \triangle ADC$.

DEMONSTRATION.



- 1 $AB \parallel CD$. Def. 35.
 2 $\angle DAB = \angle ADC$. 29.
 3 $AC \parallel BD$. Def. 35.
 4 $\angle DAC = \angle ADB$. 29.
 5 $\angle DAB + \angle DAC = \angle ADC + \angle ADB$. Ax. II.
 6 $\angle CAB = \angle CDB$.
 7 AD is common.
 8 $\angle DAB = \angle ADC$, $\angle DAC = \angle ADB$, AD is common.
 9 $AB = CD$, $AC = BD$, $\angle C = \angle B$, $\triangle ADB = \triangle ADC$. 26.
 10 $AB = CD$, $AC = BD$, $\angle C = \angle B$, $\angle CAB = \angle CDB$. $\triangle ADB = \triangle ADC$. Q. E. D.

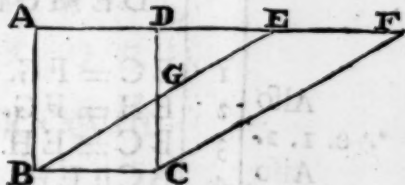
PROP.

P R O P. XXXV.

Parallelograms, $ABCD$, $BCFE$, which stand upon the same Base BC , and between the same Parallels AF , BC , are equal one to the other.

That is, $\square ABCD = \square BCFE$.

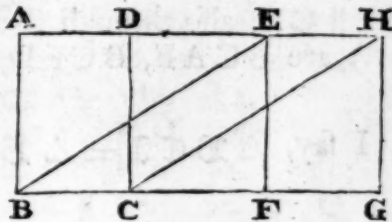
DEMONSTRATION.



- | | |
|----------------------------|---|
| 1 | $AD = BC$. 34. |
| Alfo 2 | $EF = BC$. 34. |
| $\therefore$ e. 1. 2. 3 | $AD = EF$. Ax. 1. |
| 3 + DE 4 | $AD + DE = DE + EF$. Ax. 11. |
| i. e. 5 | $AE = DF$. |
| Again 6. | $AB = DC$. 34. |
| And 7 | $\angle A = \angle CDF$. 29. |
| $\therefore$ e. 5. 6. 7. 8 | $AE = DF$, $AB = DC$, $\angle A = \angle CDF$. |
| $\therefore$ e. 8. 9 | $\triangle ABE = \triangle DCF$. 4. |
| 9 - $\triangle DGE$ 10 | $\triangle ABE - \triangle DGE = \triangle DCF - \triangle DGE$. Ax. 12. |
| i. e. 11 | Trapezium $ABGD =$ Trapezium $EGCF$. |
| 11 + $\triangle BGC$ 12 | Trap. $ABGD + \triangle BGC =$ Trap. $EGCF + \triangle BGC$. Ax. 11. |
| i. e. 13 | $\square ABCD = \square BCFE$. Q. E. D. |

P R O P. XXXVI.

Parallelograms, $ABCD$, $EFGH$, that are between the same Parallels AH , BG , and on equal Bases BC , FG , are equal.

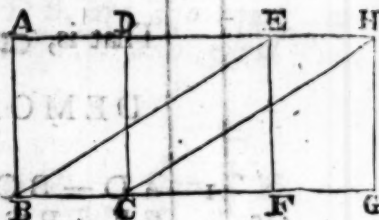


PREPARATION.

Join BE, CH.

I say, $\square ABCD = \square EFGH$.

DEMONSTRATION.



- 1 $BC = FG$. *Hyp.*
 Also 2 $EH = FG$. 34.
 $\therefore$ e. 1. 2. 3 $BC = EH$. *Ax. 1.*
 Also 4 $BC \parallel EH$. *Const.*
 $\therefore$ e. 4. 3. 5 $BC \parallel$ and $= EH$.
 $\therefore$ e. 5. 6 $BE \parallel CH$. 33.
 $\therefore$ e. 5. 6. 7 $BC \parallel EH, BE \parallel CH$.
 $\therefore$ e. 7. 8 $BCHE$ is a $\square$. *Def. 35.*
 But 9 $\square BCHE = \square ABCD$. 35.
 Also 10 $\square BCHE = \square EFGH$. 35.
 $\therefore$ e. 9. 10. 11 $\square ABCD = \square EFGH$. *Ax. 1. Q.E.D.*

PROP. XXXVII.

Triangles, BCA, BCD , standing upon the same Base BC , and between the same Parallels BC, AD , are equal one to the other.

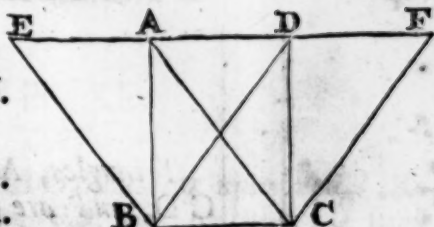
PREPARATION.

Produce AD both ways, and through B draw (31.) $BE \parallel CA$, also through C draw (31.) $CF \parallel BD$, so (Def. 35.) are $BCAE, BCFD$, Parallelograms.

I say, $\triangle BCA = \triangle BCD$.

DEMON.

DEMONSTRATION.



- 1 $\square BCFD = \square BCAE$. 35.
 2 $\square BCFD = \frac{1}{2} \square BCAE$. Ax. 14.
 Also 3 $\triangle BCA = \frac{1}{2} \square BCAE$. 34.
 $\therefore$ c. 3. 2. 4 $\triangle BCA = \frac{1}{2} \square BCFD$. Ax. 8.
 Also 5 $\triangle BCD = \frac{1}{2} \square BCFD$. 34.
 $\therefore$ c. 4. 5. 6 $\triangle BCA = \triangle EFD$. Ax. 8. Q. E. D.

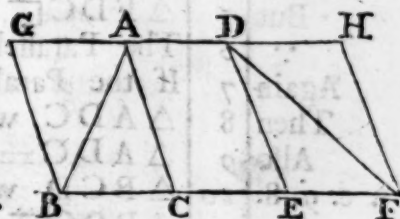
PROP. XXXVIII.

Triangles, BCA, EFD, set upon equal Bases BC, EF, and between the same Parallels AD, BF, are equal the one to the other.

PREPARATION.

Produce AD both ways, and thro' B draw (31.) BG $\parallel$ CA, also thro' F draw (31.) FH $\parallel$ ED, so (Def. 35.) are BCAG, EFHD, Parallelograms.

I say, $\triangle BCA = \triangle EFD$.



DEMONSTRATION.

- 1 $\square EFHD = \square BCAG$. 36.
 2 $\square EFHD = \frac{1}{2} \square BCAG$. Ax. 14.
 Also 3 $\triangle BCA = \frac{1}{2} \square BCAG$. 34.
 $\therefore$ c. 3. 2. 4 $\triangle BCA = \frac{1}{2} \square EFHD$. Ax. 8.
 Also 5 $\triangle EFD = \frac{1}{2} \square EFHD$. 34.
 $\therefore$ c. 4. 5. 6 $\triangle BCA = \triangle EFD$. Ax. 8. Q. E. D.

R 2

PROP.

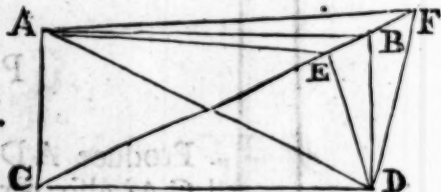
PROP. XXXIX.

Triangles, ADC, BDC, which have the same Base CD, and are equal to each other, are between the same Parallels AB, CD.

PREPARATION.

If the Parallel be said to fall above AB, through A draw (31.) $AF \parallel CD$, if below, through A draw (31.) $AE \parallel CD$, and producing CB, 'till it meet AF, join FD, ED.

I say, $AB \parallel CD$.



DEMONSTRATION.

- | | | |
|------------------------|----|---|
| For | 1 | If the Parallel should fall above AB. |
| Then | 2 | $\triangle ADC$ would be $= \triangle FDC$. 37. |
| Also | 3 | $\triangle ADC = \triangle BDC$. Hyp. |
| $\therefore$ e. 2. 3. | 4 | $\triangle FDC$ would be $= \triangle BDC$. Ax. 1. |
| But | 5 | $\triangle FDC \neq \triangle BDC$. Ax. 17. |
| $\therefore$ | 6 | The Parallel will not fall above AB. |
| Again | 7 | If the Parallel should fall below AB. |
| Then | 8 | $\triangle ADC$ would be $= \triangle EDC$. 37. |
| Also | 9 | $\triangle ADC = \triangle BDC$. Hyp. |
| $\therefore$ e. 9. 8. | 10 | $\triangle BCD$ would be $= \triangle EDC$. Ax. 1. |
| But | 11 | $\triangle BDC \neq \triangle EDC$. Ax. 17. |
| $\therefore$ | 12 | The Parallel will not fall below AB. |
| $\therefore$ e. 6. 12. | 13 | The Parallel falls neither above nor below AB. |
| $\therefore$ e. 13. | 14 | $AB \parallel CD$. Q. E. D. |

PROP.

LIX TO PROP. XL.

Equal Triangles, $\triangle AGC$, $\triangle BHD$, standing upon equal Bases CG , DH , and on the same Side, are between the same Parallels AB , CD .

PREPARATION.

If the Parallel be said to fall above AB , draw (31.) $AF \parallel CD$, if below, draw (31.) $AE \parallel CD$, and producing HE till it meet AF , join FD , ED .

I say, $AB \parallel CD$.



DEMONSTRATION.

- | | | |
|------------------------|----|---|
| For | 1 | If the Parallel should fall above AB . |
| Then | 2 | $\triangle AGC$ would be $= \triangle FHD$. 38. |
| Also | 3 | $\triangle AGC = \triangle BHD$. Hyp. |
| $\therefore$ e. 2. 3. | 4 | $\triangle FHD$ would be $= \triangle BHD$. Ax. 1. |
| But | 5 | $\triangle FHD \not\subset \triangle BHD$. Ax. 17. |
| $\therefore$ | 6 | The Parallel will not fall above AB . |
| Again | 7 | If the Parallel should fall below AB . |
| Then | 8 | $\triangle AGC$ would be $= \triangle EHD$. 38. |
| Also | 9 | $\triangle AGC = \triangle BHD$. Hyp. |
| $\therefore$ e. 9. 8. | 10 | $\triangle BHD$ would be $= \triangle EHD$. Ax. 1. |
| But | 11 | $\triangle BHD \not\subset \triangle EHD$. Ax. 17. |
| $\therefore$ | 12 | The Parallel will not fall below AB . |
| $\therefore$ e. 6. 12. | 13 | The Parallel falls neither above nor below AB . |
| $\therefore$ e. 13. | 14 | $AB \parallel CD$. Q. E. D. |

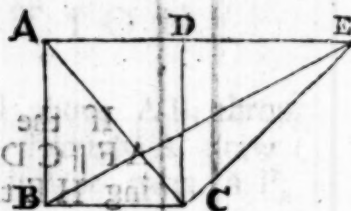
IX. P R O P. XLI.

If a Parallelogram $ABCD$, have the same Base BC , with the Triangle BCE , and be between the same Parallels AE, BC , then is the Parallelogram $ABCD$, double to the Triangle BCE .

PREPARATION.

Draw the Diagonal AC .

I say, $\square ABCD = 2 \triangle BCE$.



DEMONSTRATION.

- | | | | |
|-----------------------|---|---------------------------------------|-----------------|
| | 1 | $\triangle BCA = \triangle BCE$. | 37. |
| Also | 2 | $\triangle BCA = \triangle CDA$. | 34. |
| $\therefore$ e. 1. 2. | 3 | $\triangle BCE = \triangle CDA$. | Ax. 1. |
| 3×2 | 4 | $2 \triangle BCE = 2 \triangle CDA$. | Ax. 13. |
| Also | 5 | $\square ABCD = 2 \triangle CDA$. | 34. |
| $\therefore$ 5. 4. | 6 | $\square ABCD = 2 \triangle BCE$. | Ax. 7. Q. E. D. |

P R O P. XLII.

To make a Parallelogram $EFGC$, equal to a given Triangle ABC , and having one Angle GCE , equal to a given right-lined Angle D .

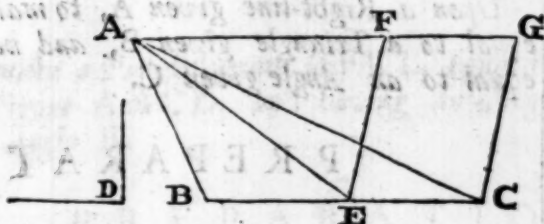
PREPARATION.

Through A draw (31.) $AG \parallel BC$; then making (23.) $\angle GCE = \angle D$, and (10.) bisecting the Base BC in E , draw AE , and (31.) $EF \parallel CG$.

I say,

I say, $\square EFGC = \triangle ABC$. $\angle GCE = \angle D$.

DEMONSTRATION.

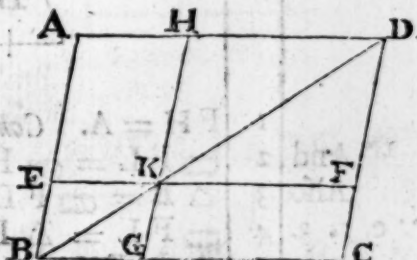


- 1 $BE = EC, AG \parallel BC$. *Const.*
 2 $\triangle ABE = \triangle AEC$. 38.
 Again 3 $\triangle ABC = \triangle ABE + \triangle AEC$. *Ax. 13.*
 4 $\triangle ABC = \triangle AEC + \triangle AEC$. *Sub.*
 i. e. 5 $\triangle ABC = 2 \triangle AEC$.
 Also 6 $\square EFGC = 2 \triangle AEC$. 41.
 7 $\square EFGC = \triangle ABC$. *Ax. 7.*
 Lastly 8 $\angle GCE = \angle D$. *Const.*
 9 $\square EFGC = \triangle ABC$. $\angle GCE = \angle D$. Q. E. F. & D.

PROP. XLIII.

In every Parallelogram ABCD, the Complements AK, KC, of those Parallelograms EG, HF, which are about the Diagonal BD, are equal.

That is, $\square AK = \square KC$.



DEMONSTRATION.

- 1 $\triangle EKB = \triangle BKG$. 34.
 Also 2 $\triangle KHD = \triangle KLD$. 34.
 1 + 2 3 $\triangle EKB + \triangle KHD = \triangle BKG + \triangle KLD$. *Ax. 11.*
 Again 4 $\triangle ABD = \triangle BDC$. 34.
 i. e. 5 $\triangle EKB + \triangle KHD + \square AK = \triangle BKG + \triangle KLD + \square KC$.
 5 - 3 6 $\square AK = \square KC$. *Ax. 12.* Q. E. D.

PROP. XLIV.

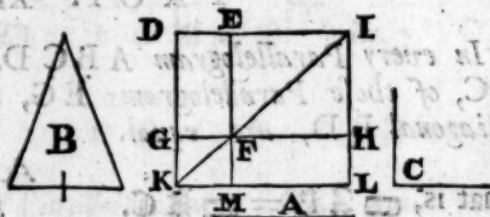
Upon a Right-line given A, to make a Parallelogram FL, equal to a Triangle given B, and having an Angle MFH, equal to an Angle given C.

PREPARATION.

Make (43.) a $\square FD = \triangle B$, so that $\angle GFE$ be $= \angle C$. Produce GF till FH be $= A$. Through H draw (31.) $IL \parallel EF$, which let DE produced meet in I. Let DG drawn forth, meet with a Right-line drawn from I, in the Point K. Through K draw (31.) $KL \parallel GH$, with which let EF drawn out meet at M, and IH at L.

I say, $\square FL = \triangle B$, $\angle MFH = \angle C$.

DEMONSTRATION



- | | | |
|--------------------------|---|--|
| | 1 | $FH = A$. <i>Const.</i> |
| And | 2 | $\square FL = \square FD$. 43. |
| Also | 3 | $\triangle B = \square FD$. <i>Const.</i> |
| $\therefore$ c. 2. 3. | 4 | $\square FL = \triangle B$. <i>Ax. 1.</i> |
| Again | 5 | $\angle MFH = \angle GFE$. 15. |
| Also | 6 | $\angle C = \angle GFE$. <i>Const.</i> |
| $\therefore$ c. 5. 6. | 7 | $\angle MFH = \angle C$. <i>Ax. 1.</i> |
| $\therefore$ c. 1. 4. 7. | 8 | $\square FL = \triangle B$, $\angle MFH = \angle C$. Q. E. F. & D. |

PROP.

PROP. XLV.

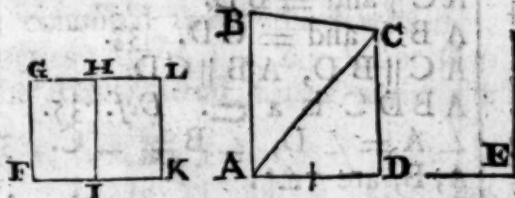
To make a Parallelogram $GFKL$, equal to a given right-lined Figure $ABCD$, and having an Angle F , equal to a given Angle E .

PREPARATION.

Resolve the right-lined Figure into Triangles ABC , ADC , and making (41.) $\square GFIH = \triangle ABC$, having $\angle F = \angle E$, produce the Line FI , and (42.) make $\square HIKL = \triangle ADC$.

I say, $\square GFKL = ABCD$. $\angle f = \angle e$.

DEMONSTRATION.



- | | | |
|-----------------------|---|---|
| | 1 | $\square GFIH = \triangle ABC$. <i>Const.</i> |
| And | 2 | $\square HIKL = \triangle ADC$. <i>Const.</i> |
| $\therefore + 2$ | 3 | $\square GFIH + \square HIKL = \triangle ABC + \triangle ADC$. <i>At 11.</i> |
| i.e. | 4 | $\square GFKL = ABCD$. |
| Again | 5 | $\angle F = \angle E$. <i>Const.</i> |
| $\therefore$ c. 4. 5. | 6 | $\square GFKL = ABCD$. $\angle f = \angle e$. Q.E.F. & D. |

PROP. XLVI.

Upon a Line given AB, to describe a Square ABDC.

PREPARATION.

Erect (11.) two Perpendiculars AC, BD, equal to AB, and join CD.

I say, ABDC is a Square.



DEMONSTRATION.

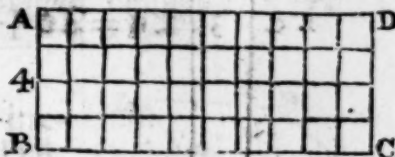
- | | | |
|---------------|----|--|
| | 1 | A, B, are $r\angle$. Def. 10. |
| ∴ e. 1. | 2 | AC $\parallel$ BD. 28. |
| Again | 3 | AC = AB = BD. Const. |
| ∴ e. 2. 3. | 4 | AC $\parallel$ and = BD. |
| ∴ e. 4. | 5 | AB $\parallel$ and = CD. 33. |
| ∴ e. 4. 5. | 6 | AC $\parallel$ BD, AB $\parallel$ CD. |
| ∴ e. 6. | 7 | ABDC is a $\square$. Def. 35. |
| ∴ e. 7. | 8 | $\angle A = \angle D$, $\angle B = \angle C$. 34. |
| But e. 1. | 9 | A, B, are $r\angle$. |
| ∴ e. 8. 9. | 10 | D, C, are $r\angle$. |
| ∴ e. 9. 10. | 11 | $\angle A = \angle B = \angle D = \angle C$. Ax. 21. |
| Also e. 3. 5. | 12 | AC = AB = BD = CD. Ax. 1. |
| ∴ e. 12. 11. | 13 | AC = AB = BD = CD, $\angle A = \angle B = \angle D = \angle C$. |
| ∴ e. 13. | 14 | ABDC is a Square. Def. 29. Q.E.F. & D. |

L.E.M.

LEMMA I.

The Area of every Square, Right Angled Parallelogram, or Rectangle, is that Space which is contain'd within its Sides, being equal to the Product of those Numbers, which express the Measure of its Sides.

Thus the Area of the Right-angled Parallelogram, or Rectangle ABCD, is that Space which is contained within its four Sides, AB, BC, CD, DA, being (if AB be = 4, BC = 10 Inches) equal to $AB \times BC, = 4 \times 10 = 40$ Square Inches.



COROLLARY.

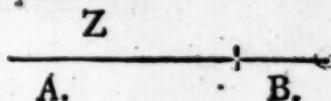
Every Right-angled Parallelogram therefore, or Rectangle ABCD, is said to be contain'd under two Right-lines AB, BC, comprehending a Right-angle B.

Wherefore, by $AB \times BC$, or such like, is always meant the Rectangle contain'd under the Right-lines, AB, BC, set at Right-angles; by $ZA = Z \times A$, or such like, is always meant the Rectangle contained under the Right-lines Z, A, set at Right-angles.

LEMMA II.

If a Right-line Z, be cut any-wise into two Parts, the Square made of the whole Z, is equal both to the Squares made of the Segments A, B, and twice a Rectangle made of the Parts A, B.

That is, $Z^2 = A^2 + B^2 + 2AB$.



DEMONSTRATION.

- 1 $Z = A + B$. Ar. 18.
 2 $ZZ = AA + BB + 2AB$. Ar. 9.
 i.e. 3 $ZQ = QQ + BQ + 2QB$. Q.E.D.

Z

B.

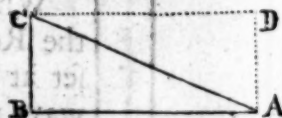
LEMMA III.

The Area of every Right-angled Triangle ABC , is equal to half the Parallelogram made of its Base AB , and Perpendicular BC .

PREPARATION.

Complete the Right-angled Parallelogram $ABCD$.

I say, $\triangle ABC = \frac{1}{2} \square ABCD$.



DEMONSTRATION.

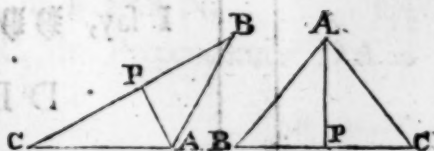
- 1 $\triangle ABC = \frac{1}{2} \square ABCD$. 34. Q.E.D.

COROLLARIES.

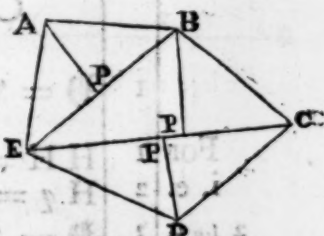
- 1 Because the Area of the $\square ABCD$ is (Lem. 1.) equal to its Base AB , drawn into its Perpendicular BC , and the $\triangle ABC$ is equal to $\frac{1}{2} \square ABCD$; therefore the Area of a Right-angled Triangle ABC , is equal to its Base AB , drawn into half its Perpendicular BC .

Hence

2 Hence the Areas of Acute and Obtuse-angled Triangles ABC , may be also found, viz. by resolving them into Right-angled Triangles, by Means of a Perpendicular AP , let fall from the Acute and Obtuse Angles upon the opposite Side. Thus the Areas of the Acute and Obtuse-angled Triangles, ABC , are $= \frac{1}{2} AP \times PB + \frac{1}{2} AP \times PC = \frac{1}{2} AP \times BC$.



3 Hence also the Area of any irregular Figure, or Field $ABCDE$, may be found, viz. by resolving it into Triangles; and each of those Triangles into Right-angled Ones, by means of Perpendiculars let fall from their vertical Angles upon their respective Bases. Thus the Area of the irregular Figure or Field $ABCDE$, is $= \frac{1}{2} AP \times BE + \frac{1}{2} BP \times CE + \frac{1}{2} DP \times CE$.

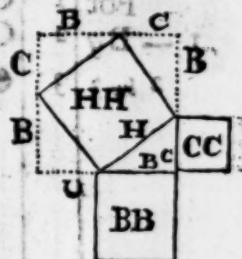


PROP. XLVII.

In any Right-angled Triangle BCH , the Square HH , describ'd upon the Hypotenuse H , is equal to the Sum of the Squares, BB , CC , described on the other two Sides B , C .

PREPARATION.

Make (46.) a Square whose Side is $= B + C$, and let that Square be express'd by S , so (Lem. 2.) will its Area be $= BB + CC + 2BC$, also the Sum of the Areas of the Four Triangles included in that Square, will (Cor. 1. Lem. 3.) be $= \frac{1}{2} BC \times 4$ $= 2BC$.



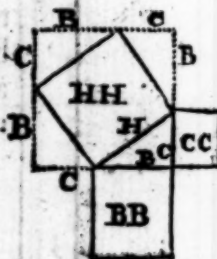
V

May,

I say, $HH = BB + CC$.

DEMONSTRATION.

1 $S = HH + 2BC$.
 Also 2 $S = BB + CC + 2BC$.
 ∴ e. i. 2. 3 $HH + 2BC = BB + CC + 2BC$. Ax. 1.
 3 - 2BC 4 $HH = BB + CC$. Ax. 12. Q. E. D.



COROLLARIES.

1 $H = \sqrt{Cq + Bq}$.
 For 1 $HH = CC + BB$.
 i. e. 2 $Hq = Cq + Bq$.
 2 w 2 3 $H = \sqrt{Cq + Bq}$. Ax. 10.
 2 $B = \sqrt{Hq - Cq}$.
 For 1 $Bq + Cq = Hq$.
 1 - Cq 2 $Bq = Hq - Cq$. Ax. 12.
 2 w 2 3 $B = \sqrt{Hq - Cq}$. Ax. 10.
 3 $C = \sqrt{Hq - Bq}$.
 For 1 $Cq + Bq = Hq$.
 1 - Bq 2 $Cq = Hq - Bq$. Ax. 12.
 2 w 2 3 $C = \sqrt{Hq - Bq}$. Ax. 10.

PROP. XLVIII.

If a Square made upon one Side BC , of a Triangle ABC , be equal to the Squares made on the other Sides AB , AC , the Angle CAB , comprehended under the other Sides AB , AC , is Right.

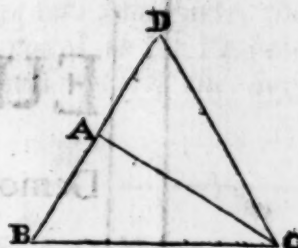
P R E.

PREPARATION.

Draw to the Point A in AC, the Perpendicular DA = AB, and joyn CD.

I say, $\angle CAB$ is a $r\angle$:

DEMONSTRATION.



- | | | |
|------------------------|----|--|
| | 1 | CAD is a $r\angle$. Def. 10. |
| $\therefore$ e. 1. | 2 | $CDq = ACq + DAq$. 47. |
| Again | 3 | $DA = AB$. Const. |
| 3 e. 2. | 4 | $DAq = ABq$. Ax. 9. |
| $\therefore$ e. 2.4 | 5 | $CDq = ACq + ABq$. Sub. |
| Also | 6 | $BCq = ACq + ABq$. Hyp. |
| $\therefore$ e. 5.6. | 7 | $CDq = BCq$. Ax. 1. |
| 7 l. 2 | 8 | $CD = BC$. Ax. 10. |
| Lastly | 9 | AC is common. |
| $\therefore$ e. 3.8.9. | 10 | $DA = AB$, $CD = BC$, AC is common. |
| $\therefore$ e. 10. | 11 | $\angle CAD = \angle CAB$. 8. |
| But e. 1. | 12 | CAD is a $r\angle$. |
| $\therefore$ e. 11.12. | 13 | $\angle CAB$ is a $r\angle$. Q. E. D. |

The End of the First Book.



EUCLID'S ELEMENTS,

Demonstrated after a New, Plain, and Easie
METHOD.

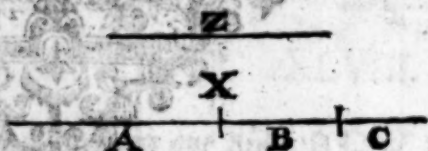
BOOK II.

PROPOSITION I.

If two Right-lines Z, X, be given, and one of them X, divided into as many Parts or Segments as you please; the Rectangle contained under the two whole Right-lines Z, X, shall be equal to all the Rectangles contained under the whole Line Z, and the several Segments A, B, C.

That is, $ZX = ZA + ZB + ZC$.

DEMONSTRATION.



$$1 \quad X = A + B + C. \text{ Ax. 18.}$$

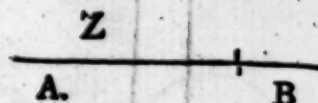
$1 \times Z$

$$2 \quad ZX = ZA + ZB + ZC. \text{ Ax. 13. Q. E. D. PROP.}$$

PROP. II.

If a Right-line Z , be divided anywise into two Parts, the Square made of the whole Line Z , is equal to the Rectangles contained under the whole Line Z , and each of the Segments A , B .

That is, $Zq = Za + Zb$.



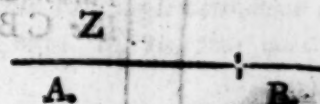
DEMONSTRATION.

- 1 $Z = A + B$. Ax. 18.
 2 $ZZ = ZA + ZB$. Ax. 13.
 i.e. 3 $Zq = Za + Zb$. Q.E.D.

PROP. III.

If a Right-line Z , be divided anywise into two Parts, the Rectangle contained under the whole Line Z , and one of the Segments B , is equal to the Rectangle made of the Segments A , B , and the Square described on the said Segment B .

That is, $Zb = Ab + Bq$.



DEMONSTRATION.

- 1 $Z = A + B$. Ax. 18.
 2 $Zb = Ab + Bb$. Ax. 13.
 i.e. 3 $Zb = Ab + Bq$. Q.E.D.

PROP.

P R O P. IV.

If a Right-line Z , be cut anywise into two Parts, the Square made of the whole Z , is equal both to the Squares made of the Segments A , B , and twice a Rectangle made of the Parts A , B .



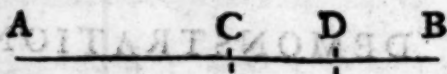
That is, $Zq = Aq + Bq + 2AB$.

D E M O N S T R A T I O N.

1 $Zq = Aq + Bq + 2AB$. Lem. 2. 1. Q. E. D.

P R O P. V.

If a Right-line AB , be cut equally in C , and unequally in D , the Rectangle contained under the unequal Parts AD , DB , together with the Square that is made of the Difference of the Parts CD , is equal to the Square that is made of the Half-line CB .



That is, $AD \times DB + CDq = CBq$.

D E M O N S T R A T I O N.

1 $CDq + DBq + CD \times DB \times 2 = CBq$. 4
 And 2 $CB \times DB = CD \times DB + DBq$. 3.
 ∴ e. 1. 2. 3 $CDq + CB \times DB + CD \times DB = CBq$. Sub.
 Again 4 $AC = CB$. Hyp.
 ∴ e. 3. 4. 5 $CDq + AC \times DB + CD \times DB = CBq$. Sub.
 Lastly 6 $AD \times DB = AC \times DB + CD \times DB$. 1.
 ∴ e. 5. 6. 7 $AD \times DB + CDq = CBq$. Sub. Q. E. D.

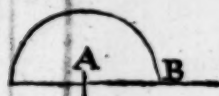
P R O P.

PROP. VI.

If a Right-line A, be divided into two equal Parts, and another Right-line B, added to the same directly in one Right-line; Then the Rectangle contained under the whole and the Line added, viz. A + B, and the Line added B, together with the Square which is made of one half the Line A, is equal to the Square of $\frac{1}{2}A + B$, taken as one Line.

That is, $\overline{A+B} \times B + \frac{1}{4}\overline{AA} = \overline{\frac{1}{2}A+B}^2$

DEMONSTRATION.

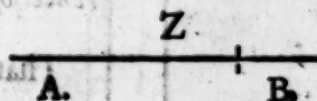


- | | | | |
|----------------------------|---|--|-----------------|
| | 1 | $AB + BB = \overline{A+B} \times B.$ | |
| $\frac{1}{4}\overline{AA}$ | 2 | $AB + BB + \frac{1}{4}\overline{AA} = \overline{A+B} \times B + \frac{1}{4}\overline{AA}.$ | AX. II. |
| Also | 3 | $\overline{A+B} \times B + \frac{1}{4}\overline{AA} = \overline{\frac{1}{2}A+B}^2.$ | |
| $\therefore c. 2. 3.$ | 4 | $\overline{A+B} \times B + \frac{1}{4}\overline{AA} = \overline{\frac{1}{2}A+B}^2$ | AX. I. Q. E. D. |

PROP. VII.

If a Right-line Z, be divided anywise into two Parts, the Square made of the whole Line Z, together with the Square made of one of the Segments B, is equal to a double Rectangle contained under the whole Line Z, and the said Segment B, together with the Square made of the other Segment A.

That is, $Zq + Bq = 2ZB + Aq.$



DEMONSTRATION.

- | | | | |
|-----------------------|---|-----------------------------|----------|
| | 1 | $ZB = AB + Bq.$ | 3. |
| 1×2 | 2 | $2ZB = 2AB + 2Bq.$ | AX. 13. |
| $2 - 2Bq$ | 3 | $2ZB - 2Bq = 2AB.$ | AX. 12. |
| Again | 4 | $Zq = Aq + Bq + 2AB.$ | 4. |
| $\therefore c. 3. 4.$ | 5 | $Zq = Aq + Bq + 2ZB - 2Bq.$ | Sub. |
| i. e. | 6 | $Zq = Aq + 2ZB - Bq.$ | |
| $6 + Bq$ | 7 | $Zq + Bq = 2ZB + Aq.$ | Q. E. D. |

X 2

PROP.

PROP. VIII.

If a Right-line Z, be divided anywise into two Parts, the Rectangle contained under the whole Line Z, and one of the Segments B four Times, together with the Square of the other Segment A, is equal to the Square of the whole Line Z, and the other Segment B, viz. of $Z + B$, taken as one Line.

That is, $4ZB + Aq = Z + B^2$

Z

DEMONSTRATION.

A.

B

1 + 2ZB

Allo

∴ e. 2. 3.

$Zq + Bq = 2ZB + Aq$ 7.
 $Zq + Bq + 2ZB = 4ZB + Aq$ Ax. 11.
 $Zq + Bq + 2ZB = Z + B^2$ Ax. 1. Q. E. D.

PROP. IX.

If a Right-line AB, be cut equally in C, and unequally in D, the Squares of the unequal Parts AD, DB, taken together, are double to the Squares of the Half-line AC, and the Difference of the Parts CD.

That is, $ADq + DBq = 2ACq + 2CDq$

DEMONSTRATION.

A C D B

ADq

- 1 $ADq = ACq + CDq + \overline{AC \times CD} \times 2. 4.$
 2 $ADq + DBq = ACq + CDq + \overline{AC \times CD} \times 2 + DBq. Ax. 11.$
 Again 3 $AC = CB. Hyp.$
 4 $ADq + DBq = ACq + CDq + \overline{CB \times CD} \times 2 + DBq. Sub.$
 Lastly 5 $\overline{CB \times CD} \times 2 + DBq = CBq + CDq. 7.$
 6 $ACq = CBq. Ax. 9.$
 7 $\overline{CB \times CD} \times 2 + DBq = ACq + CDq. Sub.$
 8 $ADq + DBq = ACq + CDq + ACq + CDq. Sub.$
 9 $2ADq + 2DBq = 2ACq + 2CDq. Q. E. D.$

PROP. X.

If a Right-line A, be divided into two equal Parts, and another Line be added in a Right-line with the same; then is the Square of the whole Line, together with the added Line, as being one Line, together with the Square of the added Line B, double to the Square of half A, and the added Line B, taken as one Line, and to the Square of $\frac{1}{2} A$.

That is, $\overline{A + B}^2 + Bq = \overline{\frac{1}{2}A + B}^2 \times 2 + \frac{1}{2}AA.$

DEMONSTRATION.



- 1 $\overline{A + B}^2 = Aq + Bq + 2AB.$
 2 $\overline{A + B}^2 + Bq = Aq + 2Bq + 2AB. Ax. 11.$
 Again 3 $\frac{1}{2}\overline{A + B}^2 = \frac{1}{2}AA + Bq + AB.$
 4 $\frac{1}{2}\overline{A + B}^2 \times 2 = \frac{1}{2}AA + 2Bq + 2AB. Ax. 13.$
 5 $\frac{1}{2}\overline{A + B}^2 \times 2 + \frac{1}{2}AA = Aq + 2Bq + 2AB. Ax. 11.$
 6 $\overline{A + B}^2 + Bq = \overline{\frac{1}{2}A + B}^2 \times 2 + \frac{1}{2}AA. Q. E. D.$

PROP.

PROP. XI.

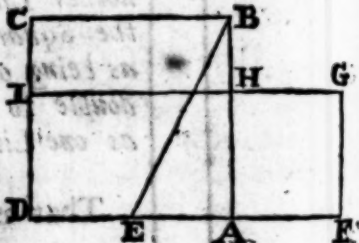
To cut a Right-line AB, in the Point H, so that the Square of one of its Parts AH, may be equal to the Rectangle contained under the whole Line AB, and the other Part BH.

PREPARATION.

Upon AB describe (46. 1.) the Square ABCD. Bisect (10. 1.) DA in E, and draw EB. From the Line DA produced take EF = EB. On AF make (46. 1.) the Square AG, and extend GH to I

I say, $AH^2 = AB \times BH$.

DEMONSTRATION.



- | | |
|----------------------------|--------------------------------------|
| 1 | $DF \times AF + EAq = EFq$. 6. |
| And 2 | $AF = FG$. Def. 29. 1. |
| $\therefore$ c. 1. 2. 3 | $DF \times FG + EAq = EFq$. Sub. |
| i. c. 4 | $DG + EAq = EFq$. |
| Again 5 | $EB = EF$. Const. |
| $\therefore$ c. 2. 6 | $EBq = EFq$. Ax. 9. |
| $\therefore$ c. 4. 6. 7 | $DG + EAq = EBq$. Sub. |
| Also 8 | $EAq + ABq = EBq$. 47. 1. |
| $\therefore$ c. 7. 8. 9 | $DG + EAq = EAq + ABq$. Ax. 1. |
| 9 - EAq 10 | $DG = ABq$. Ax. 12. |
| 10 - AI 11 | $DG - AI = ABq - AI$. Ax. 12. |
| i. c. 12 | $AHq = BI$. |
| Also 13 | $BC \times BH = BI$. |
| $\therefore$ c. 12. 13. 14 | $AHq = BC \times BH$. Ax. 1. |
| Lastly 15 | $AB = BC$. Def. 29. 1. |
| $\therefore$ c. 14. 15. 16 | $AHq = AB \times BH$. Sub. Q. E. D. |

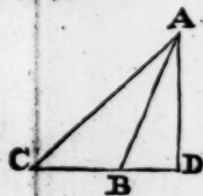
PROP.

P R O P. XII.

In Obtuse-angled Triangles ABC , the Square that is made of the Side AC , subtending the Obtuse Angle ABC , is greater than the Squares of the Sides AB , CB , that contain the Obtuse-Angle ABC , by a double Rectangle contained under one of the Sides CB , which are about the Obtuse-Angle ABC , on which Side produced the Perpendicular AD falls, and under the Line BD , taken without the Triangle from the Point on which the Perpendicular AD falls, to the Obtuse-Angle ABC .

That is, $AC^2 = AB^2 + CB^2 + CB \times BD \times 2$.

DEMONSTRATION.



1. $AC^2 = AD^2 + CD^2$. 47. I.
 And 2. $CD^2 = BD^2 + CB^2 + CB \times BD \times 2$. 4.
 ∴ e. 1. 2. 3. $AC^2 = AD^2 + BD^2 + CB^2 + CB \times BD \times 2$. Sub.
 Again 4. $AB^2 = AD^2 + BD^2$. 47. I.
 ∴ e. 3. 4. 5. $AC^2 = AB^2 + CB^2 + CB \times BD \times 2$. Sub. Q. E. D.

P R O P. XIII.

In Acute-angled Triangles ABC , the Square made of the Side AB , subtending the Acute-angle ACB , is less than the Squares made on the Sides AC , BC , comprehending the Acute-angle ACB , by a double Rectangle contained under one of the Sides BC , which are about the Acute-angle ACB , on which the Perpendicular AD falls, and under the Line DC , taken within the Triangle, from the Perpendicular AD , to the Acute-angle ACB .



Y. 2.

That

That is, $AB^2 + BC \times DC \times 2 = AC^2 + BC^2$.

DEMONSTRATION.

- 1 $AD^2 + DC^2 = AC^2$. 47. 1.
 2 $AD^2 + BC^2 + DC^2 = AC^2 + BC^2$. Ar. 11.
 3 Again $BC^2 + DC^2 = BC \times DC \times 2 + BD^2$. 7.
 4 $AD^2 + BD^2 + BC \times DC \times 2 = AC^2 + BC^2$. Sub.
 5 Lastly $AB^2 = AD^2 + BD^2$. 47. 1.
 6 $AB^2 + BC \times DC \times 2 = AC^2 + BC^2$. Sub. Q. E. D.



PROP. XIV.

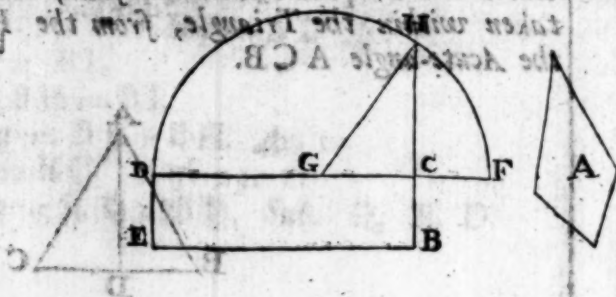
To find a Right-line CH, whose Square is equal to a Right-lined Figure given A.

PREPARATION.

Make (45. 1.) the Rectangle $DB = A$, and produce the greater Side thereof DC to F , so that CF be $= BC$. Bisect (10. 1.) DF in G , about which as the Center at the Distance of GF , describe the Circle FHD , and drawing out BC , till it touch the Circumference in H , joyn GH .

I say, $CH^2 = A$.

DEMONSTRATION.



DC

	1	$DC \times CF + GCq = GFq.$ 5.
And	2	$CF = BC.$ <i>Const.</i>
$\therefore$ e. 1. 2.	3	$DC \times BC + GCq = GFq.$ <i>Sub.</i>
i. e.	4	$DB + GCq = GFq.$
Again	5	$GH = GF.$ <i>Def. 16. 1.</i>
$\therefore$ e. 2.	6	$GHq = GFq.$ <i>Ax. 9.</i>
$\therefore$ e. 4. 6.	7	$DB + GCq = GHq.$ <i>Ax. 1.</i>
Also	8	$GCq + CHq = GHq.$ 47. 1.
$\therefore$ e. 7. 8.	9	$DB + GCq = GCq + CHq.$ <i>Ax. 1.</i>
$\therefore$ GCq.	10	$DB = CHq.$ <i>Ax. 12.</i>
Also	11	$DB = A.$ <i>Const.</i>
$\therefore$ e. 10. 11.	12	$CHq = A.$ <i>Ax. 1. Q. E. F. & D.</i>

BOOK III.

The End of the Second Book.

DEFINITIONS



EUCLID'S

2

EUCLID'S ELEMENTS,

Demonstrated after a New, Plain, and Easie

METHOD.

BOOK III.

DEFINITIONS.

1



Equal Circles are such, whose Diameters or Semi-diameters are equal.

2

A Line is said to touch a Circle, when, meeting with its Circumference, it does not cut it.

As the Line A B.

The Right-line C B, cuts the Circle.



3

Circles touch, when meeting they do not cut each other.

As the Circles A, B.

The Circle C, cuts the Circle B.



In

4 In a Circle $ABCD$, Right-lines AB , DC , are said to be equally distant from the Center, when Perpendiculars EF , EH , drawn from the Center E to them, are equal.



5 A Segment of a Circle, is a Figure terminated on one Side by a Right-line, and on the other by the Circumference of a Circle.

As ABC , ADC .



6 An Angle is said to be in a Segment, when in the Circumference thereof, some Point B is taken, and from it Right-lines BA , BC , are drawn to its Base AC .

As the Angle ABC .



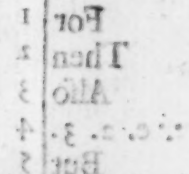
7 An Angle is upon that Arch, to which it is opposed, or which is as its Base.

As the Angle ABC , is upon the Arch AC .



8 A Sector, is a Figure contained under two Semidiameters, and the Arch which serves them for a Base.

As the Figure BCA .



9 Similar Segments, are those that include equal Angles; or, in whom the Angles ABC , DEF , are equal.

As the Segments ABC , DEF .



PROPOSITION I.

To find the Center F, of a given Circle ABC.

PREPARATION.

Draw a Right-line AC, anywise in the Circle ABC, which (10. 1.) bisect in E; through E draw (11. 1.) a Perpendicular BD, and bisect the same in F, so will the Point F be the Center.

If the Center be said not to be in BD, and consequently not in its Center F, suppose it in the Point G without BD; and draw GA, GE, GC; so (GC being (Def. 16. 1.) = GA, EC (Const.) = EA, and GE common) will $\angle GEC$ be (8. 1.) = $\angle GEA$, and consequently (Def. 10. 1.) Right.



I say, F is the Center.

DEMONSTRATION.

- | | | |
|------------|---|---|
| For | 1 | If the Center should be in the Point G, without BD. |
| Then | 2 | $\angle GEC$ would be a r $\angle$. |
| Also | 3 | $\angle FEC$ is a r $\angle$. Def. 10. 1. |
| ∴ c. 2. 3. | 4 | $\angle GEC$ would be = $\angle FEC$. Ax. 21. |
| But | 5 | $\angle GEC \neq \angle FEC$. Ax. 17. |
| ∴ | 6 | The Center is not in the Point G, without BD. |
| And | 7 | Since for the same Reason it is in no other Point without BD. |
| ∴ | 8 | The Center is in BD. |
| ∴ | 9 | F is the Center. Q. E. F. & D. |

COROL.

COROLLARY.

The Center of a Circle is in that Line, which falling perpendicularly upon another, divides it into two equal Parts.

PROP. II.

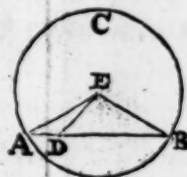
If in the Circumference of a Circle ABC, any two Points A, B, be taken, the Right-line AB, which joins those two Points, shall fall within the Circle.

PREPARATION.

Take, in the Right-line AB, any Point D; and from the Center E, draw EA, ED, EB.

I say, AB falls within the Circle.

DEMONSTRATION.



- | | | |
|-------------|---|---------------------------------------|
| | 1 | EA = EB. Def. 16. 1. |
| v. c. 1. | 2 | $\angle A = \angle B.$ 5. 1. |
| Again | 3 | $\angle EDB \square \angle A.$ 16. 1. |
| v. c. 2. 3. | 4 | $\angle EDB \square \angle B.$ Ax. 2. |
| v. c. 4. | 5 | EB $\square$ ED. 19. 1. |
| But | 6 | EB only reaches the Circumference. |
| v. c. 5. 6. | 7 | ED comes not so far. |
| v. c. 7. | 8 | The Point D falls within the Circle. |
| v. c. 8. | 9 | AB falls within the Circle. Q. E. D. |

COROLLARY.

A Right-line doth not touch the Circumference of a Circle, but in one Point; because if it should touch it in two, it might be drawn from one of those Points to the other, and so would enter within the Circle, and consequently cut its Circumference, and not touch it.

A a

PROP.

PROP. III.

If in a Circle $ABCD$, a Right-line BD , drawn thro' the Center, bisect any other Line AC , not drawn thro' the Center E , it shall cut it at Right-angles; and if it cut it at Right-angles, it shall also bisect the same.

Hyp. 1 Suppose BD to bisect AC .

PREPARATION.

Draw EA, EC .

I say, $\angle EFA, \angle EFC$ are $\angle$'s.

DEMONSTRATION.

- 1 $FA = FC$. Hyp.
 And 2 $EA = EC$. Def. 16. 1.
 And 3 EF is common.
 4 $FA = FC, EA = EC, EF$ is common.
 5 $\angle EFA = \angle EFC$. 8. 1.
 6 $\angle EFA, \angle EFC$ are $\angle$'s. Def. 16. 1. Q. E. D.



Hyp. 2 Suppose BD to cut AC at Right-angles.

I say, $FA = FC$.

DEMON

DEMONSTRATION.

- 1 $\angle EFA = \angle EFC$. Ax. 21.
 And 2 $EA = EC$. Def. 16. 1.
 $\therefore$ e. 2. 3 $\angle A = \angle C$. 5. 1.
 Again 4 EF is common.
 $\therefore$ e. 1. 3. 4. 5 $\angle EFA = \angle EFC$, $\angle A = \angle C$, EF is common;
 $\therefore$ e. 5. 6 $FA = FC$. 26. 1. Q. E. D.

PROP. IV.

If in a Circle $ABCD$, two Right-lines AC , BD , not drawn thro' the Center F , cut each other in a Point E , those two Lines AC , BD , do not cut each other into two equal Parts.

PREPARATION.

Join EF .

I say, AC , BD , do not cut each other into two equal Parts.

DEMONSTRATION.



- For 1 AC , BD , should cut each other into two equal Parts..
 Then 2 FEB , FEA , would be $\angle$'s. 3.
 $\therefore$ e. 2. 3 $\angle FEB$ would be $= \angle FEA$. Ax. 21.
 But 4 $\angle FEB < \angle FEA$. Ax. 17.
 $\therefore$ 5 AC , BD , do not cut each other into two equal Parts. Q. E. D.

A 2 2

PROP.

PROP. V.

If two Circles, BAC, BDC, cut one the other, they shall not have the same common Center. E.

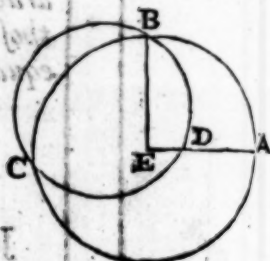
PREPARATION.

From the Center E, of the Circle BAC, draw EA, EB.

I say, E is not the common Center.

DEMONSTRATION.

- | | | |
|--------------|---|--------------------------------------|
| For | 1 | If E should be the common Center. |
| Then | 2 | EB would be = ED. Def. 16. 1. |
| Also | 3 | EB = EA. Def. 16. 1. |
| e. 3. 2. | 4 | EA would be = ED. Ax. 1. |
| But | 5 | EA $\neq$ ED. Ax. 17. |
| $\therefore$ | 6 | E is not the common Center. Q. E. D. |



PROP. VI.

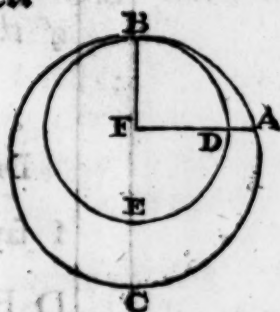
If two Circles, BAC, BDE, touch one another inwardly in B, they shall not have the same common Center F.

PREPARATION.

From the Center F, of the Circle BDE, draw FB, FD, and produce FD to A.

I say, F is not the common Center.

DEMONSTRATION.



For 1 If F should be the common Center.

Then 2 FB would be $= FA$. Def. 16. 1.

Also 3 $FB = FD$. Def. 16. 1.

v.e. 2. 3. 4 FA would be $= FD$. Ax. 1.

But 5 $FA < FD$. Ax. 17.

∴ 6 F is not the common Center. Q. E. D.

PROP. VII.

If many Lines be drawn from any one Point C , within a Circle $AFBG$, which is not its Center D , to its Circumference: 1st. That Line CB , which passes thro' its Center, is the greatest: 2d. The Remainder of it CA , continued to the opposite Part of the Circumference, is the least: 3d. That Line CE , which is nearest the greatest CB , exceeds those that are more remote: 4th. There can be no more than two of them equal to each other.

1st. That Line CB , which passes thro' its Center D , is the greatest.

PREPARATION.

Draw DE .

I say, for Example, $CD < CE$.

DEMONSTRATION.



1 $CD + DE < CE$. 20. 1.

And 2 $DB = DE$. Def. 16. 1.

v.e. 1. 2. 3 $CD + DB < CE$. Sub.

i.e. 4 $CB < CE$. Q. E. D.

B D

The

2d. The Remainder of it CA, continued to the opposite Part of the Circumference, is the least.

PREPARATION.

Draw CF, DF.

I say, for Example, $\angle ACF$.

DEMONSTRATION.

- 1 $DF \sqsubset CD + CF$. 20. 1.
 And 2 $DA = DF$. Def. 16. 1.
 $\therefore$ e. 1. 2. 3 $DA \sqsubset CD + CF$. Sub.
 3 - CD 4 $DA - CD \sqsubset CF$. Ax. 16.
 i.e. 5 $\angle ACF$. Q. E. D.



3d. That Line CE, which is nearest the greatest CB, exceeds those that are more remote.

That is, for Example, $\angle CEF$.

DEMONSTRATION.

- 1 $DE = DF$. Def. 16. 1.
 And 2 CD is common.
 And 3 $\angle CDE \sqsubset \angle CDF$. Ax. 17.
 $\therefore$ e. 1. 2. 3. 4 $DE = DF$, CD is common, $\angle CDE \sqsubset \angle CDF$.
 $\therefore$ e. 4. 5 $\angle CEF$. 24. 1. Q. E. D.

4th. There can be no more than two of them equal to each other.

PREPARATION.

Make (23. 1.) at the Center D, $\angle CDG = \angle CDF$,
 and join CG. I say,

I say, There can be no more than two of them equal to each other.

DEMONSTRATION.

- 1 $DG = DF$. Def. 16. 1.
 And 2 CD is common, $\angle CDG = \angle CDF$. Const.
 ∴ e. 1. 2. 3 $DG = DF$, CD is common, $\angle CDG = \angle CDF$.
 ∴ e. 3. 4 $CG = CF$. 4. 1.
 But 5 All other Lines will be either nearer or further off CB .
 ∴ 6 All other Lines will be either $\square$ or $\rhd$ CG , and CF . 3d. 7.
 ∴ 7 There can be no more than two of them equal to each other. Q. E. D.

PROP. VIII.

If from a Point C , taken without a Circle $AGBF$, many Lines be drawn to its Circumference: 1st. Of all those that extend to the Concave Circumference, that Line CB , which passes thro' the Center D , is the greatest: 2d. That Line CE , which is nearest the greatest CB , exceeds those that are more remote: 3d. Among those that fall upon the Convex Circumference, that Line CA , which being continued passes thro' the Center, is the least: 4th. That Line CI , which is nearer CA , is less than those which are further off: 5th. There can be but two equal Lines drawn, from the same Point, either to the Concave or Convex Circumference.



Of all those that extend to the Concave Circumference, that Line CB, which passes thro' the Center D, is the greatest.

PREPARATION.

Draw DE.

I say, for Example, $\angle CDE$.

DEMONSTRATION.

1 $CD + DE \leq CE$. 29. I.

2 $DB = DE$. Def. 16. I.

3 $CD + DB \leq CE$. Sub.

4 $\angle CDE$. Q. E. D.

That Line CE, which is nearest the greatest CB, exceeds those that are more remote.

PREPARATION.

Draw CF, DF.

I say, for Example, $\angle CEF$.

DEMONSTRATION.

1 $DE = DF$. Def. 16. I.

And 2 CD is common.

And 3 $\angle CDE \leq \angle CDF$. Ax. 17.

∴ e. 1. 2. 3. 4 $DE = DF$, CD is common, $\angle CDE \leq \angle CDF$.

∴ e. 4. 5 $\angle CEF$. 24. I. Q. E. D.



Among

3d. Among those that fall upon the Convex Circumference, that Line CA, which being continued passes through the Center, is the least.

PREPARATION.

Draw DI.

I say, for Example, $CA < CI$.

DEMONSTRATION.

- 1 $CD < CI + DI$ 20. 1.
- i.e. 2 $CA + AD < CI + DI$
- Again 3 $DI = AD$ Def. 16. 1.
- i.e. 2. 3. 4 $CA + AD < CI + AD$ Sub.
- 4 - AD 5 $CA < CI$ Ax. 16. Q. E. D.

4th. That Line CI, which is nearer CA, is less than those which are further off.

PREPARATION.

Draw DH.

I say, for Example, $CI < CH$.

DEMONSTRATION.

- 1 $CI + DI < CH + DH$ 21. 1.
- And 2 $DI = DH$ Def. 16. 1.
- i.e. 1. 2. 3 $CI + DH < CH + DH$ Sub.
- 3 - DH 4 $CI < CH$ Ax. 16. Q. E. D.

5th. There can be but two equal Lines drawn, from the same Point, either to the Concave or Convex Circumference.

C c

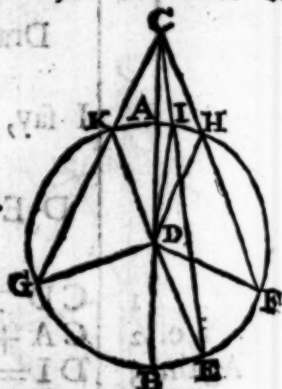
P R E



PREPARATION.

Make (13. 1.) at the Center D, $\angle CDG = \angle CDF$, $\angle CDK = \angle CDH$, and draw CKG.

I say, There can be but two equal Lines drawn, from the same Point, either to the Concave or Convex Circumference.



DEMONSTRATION.

- 1 $DG = DF$. Def. 16. 1.
 And 2 CD is common, $\angle CDG = \angle CDF$. Const.
 3 $DG = DF$, CD is common, $\angle CDG = \angle CDF$.
 4 $CG = CF$. 4. 1.
 Again 5 $DK = DH$. Def. 16. 1.
 And 6 CD is common, $\angle CDK = \angle CDH$. Const.
 7 $DK = DH$, CD is common, $\angle CDK = \angle CDH$.
 8 $CK = CH$. 4. 1.
 9 $CG = CF$, $CK = CH$.
 But 10 All other Lines will be either nearer or more remote from CB and CA .
 11 All other Lines will be either $\lrcorner$ or $\rceil$ CG and CF , CK and CH . 2d and 4th. 18.
 12 There can be but two equal Lines drawn, from the same Point, either to the Concave or Convex Circumference. Q. E. D.

P R O P. IX.

That Point, from whence three equal Lines AB , AC , AD , can be drawn to the Circumference of a Circle $ABCD$, is its Center. That

That is, *A* is its Center.

DEMONSTRATION.



More than two equal Right-lines cannot be drawn from a Point out of the Center to the Circumference. 4th. 7.

But $AB = AC = AD$. Hyp.
 $\therefore$ *A* is its Center. Q. E. D.

PROP. X.

Two Circles *AEBD*, *ABDF*, cannot cut each other in more than two Points.

PREPARATION.

If *AEBD*, *ABDF*, be said to cut each other in more than two Points *A*, *B*, *D*; find (1) the Center *C* of the Circle *AEBD*; and draw *CA*, *CB*, *CD*, which (Def. 16. 1.) will be equal the one to the other.

I say, *AEBD*, *ABDF*, cannot cut each other in more than two Points.

DEMONSTRATION.



For 1 If *AEBD*, *ABDF*, should cut each other in more than two Points.

Then 2 *C* would be the Center of the Circle *ABDF*. 9.

Also 3 *C* is the Center of the Circle *AEBD*.

$\therefore$ 2. 3. 4 Two Circles which cut each other, would have the same common Center.

But 5 Two Circles which cut each other, have not the same common Center. 5.

$\therefore$ 6 *AEBD*, *ABDF*, cannot cut each other in more than two Points. Q. E. D.

C. C. 2.

PROP.



PROP. XI.

If two Circles, ABC, ADE, touch each other within, at the Point A, the Right-line drawn thro' their Centers, being produced, will pass thro' the Point of Contact.

PREPARATION.

Draw GA, GD, FA.

I say, The Right-line drawn thro' their Center will pass thro' the Point of Contact.

DEMONSTRATION.

For if the Points G, F, were their Centers, so that the Right-line drawn thro' them should pass thro' other Points.

Then GA would be = GD, FA would be = FB.

Def. 16. 1.

Now GA = GF + FA. 20. 1.

GD would be = GF + FB. Sub.

But GD = GF + FB. Ax. 17.

The Right-line drawn thro' their Centers will not pass thro' other Points.

The Right-line drawn thro' their Centers will pass thro' the Point of Contact. Q. E. D.



PROP.

P R O P. XII.

If two Circles ABC, DBE, touch each other without, at the Point B, the Right-line drawn thro' their Centers will pass thro' the Point of Contact.

P R E P A R A T I O N.

Draw GB, HB, GH.

I say, The Right-line drawn thro' their Centers will pass thro' the Point of Contact.

D E M O N S T R A T I O N.



- | | | |
|--------------|---|---|
| For | 1 | If the Points G, H, were their Centers, so that the Right-line drawn thro' them should pass thro' other Points. |
| Then | 2 | GA would be = GB, HD would be = HB. Def. 16. 1. |
| Now | 3 | GH $\sqsupset$ GB + HB. 20. 1. |
| c. 2. 3 | 4 | GH would be $\sqsupset$ GA + HD. Sub. |
| But | 5 | GH $\sqsubset$ GA + HD. Ax. 17. |
| $\therefore$ | 6 | The Right-line-drawn thro' their Centers will not pass thro' other Points. |
| $\therefore$ | 7 | The Right-line drawn thro' their Centers will pass thro' the Point of Contact. Q. E. D. |

P R O P. XIII.

Two Circles ABC, ABD, touch each other only in one Point, whether it be within or without.

Hyp. 1 Suppose them to touch within, at the Point A.

P R E P A R A T I O N.

Having drawn thro' their Centers E, F, the Right-line EF, which being produced, will (11.) pass thro' the Point of Contact; draw the Right-lines EB, FB.

I say, They touch only in one Point.

D E M O N S T R A T I O N.



For	1	If they should touch in another Point; as B.
Then	2	EA would be = EB. Def. 16. 1.
Now	3	EB + EF < FB. 20. 1.
∴ e. 2. 3.	4	EA + EF would be < FB. Sub.
But	5	EA + EF = FB. Def. 16. 1.
∴	6	They do not touch in another Point; as B.
∴	7	They touch only in one Point. Q. E. D.

Hyp. 2 Suppose them to touch without, at the Point A.

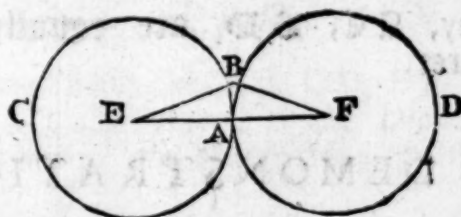
P R E-

PREPARATION.

Having drawn thro' their Centers E, F, the Right-line EF, which (12.) will pass thro' the Point of Contact ; draw the Right-lines EB, FB.

I say, They touch only in one Point.

DEMONSTRATION.



- | | | |
|----------|---|---|
| For | 1 | If they should touch in another Point ; as B. |
| Then | 2 | EA would be = EB, FA would be = FB. Def. 16. 1. |
| Now | 3 | EB + FB $\square$ EF. 20. 1. |
| e. 2. 3. | 4 | EA + FA would be $\square$ EF. Sub. |
| But | 5 | EA + FA = EF. |
| ∴ | 6 | They do not touch in another Point ; as B. |
| ∴ | 7 | They touch only in one Point. Q. E. D. |

P R O P. XIV.

In a Circle ABDC, equal Right-lines AC, BD, are equally distant from the Center ; and Right-lines AC, BD, equally distant from the Center, are equal.



Suppose.

Hyp. 1

Suppose $AC = BD$.

PREPARATION.

From the Center E, draw the Perpendiculars EF, EG, which (3.) will bisect the Lines AC, BD, and join EA, EB.

I say, AC , BD , are equally distant from the Center.

DEMONSTRATION.



- | | | |
|---------|---|--|
| | 1 | $AC = BD$. Hyp. |
| ∴ c. 1. | 2 | $AF = BG$. Ax. 8. |
| Again | 3 | $EA = EB$. Def. 16. 1. |
| 2 c. 2. | 4 | $AFq = BGq$. Ax. 9. |
| 3 c. 2. | 5 | $EAq = EBq$. Ax. 9. |
| 5 — 4. | 6 | $EAq - AFq = EBq - BGq$. Ax. 12. |
| i. c. | 7 | $EFq = EGq$. 47. 1. |
| 7 l. 2. | 8 | $EF = EG$. Ax. 10. |
| ∴ c. 8. | 9 | AC , BD , are equally distant from the Center.
Def. 4. Q. E. D. |

Hyp. 2

Suppose AC , BD , equally distant from the Center.

I say, $AC = BD$.

DEMON.

DEMONSTRATION.

- | | | |
|----------|---|----------------------------------|
| | 1 | $EF = EG.$ Def. 4. |
| And | 2 | $EA = EB.$ Def. 16. 1. |
| 1. 9. 2. | 3 | $EFq = EGq.$ Ax. 9. |
| 2. 9. 2. | 4 | $EAq = EBq.$ Ax. 9. |
| 4. — 3. | 5 | $EAq - EFq = EBq - EGq.$ Ax. 12. |
| i. e. | 6 | $AFq = BGq.$ 47. 1. |
| 6. 1. 2. | 7 | $AF = BG.$ Ax. 10. |
| i. e. 7. | 8 | $AC = BD.$ Ax. 7. Q. E. D. |

PROP. XV.

If several Right-lines AB, CD, EF, be drawn in a Circle AHBG, the greatest is the Diameter AB, and that which is nearest the Center N, is greater than that which is further off.

The greatest is the Diameter AB.

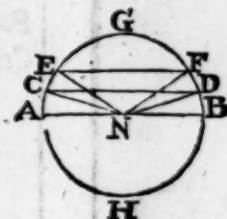
PREPARATION.

Draw NC, ND.

I say, for Example, $AB \sqsupset CD.$

DEMONSTRATION.

- | | | |
|----------|---|--------------------------------|
| | 1 | $NC = NA.$ Def. 16. 1. |
| And | 2 | $ND = NB.$ Def. 16. 1. |
| 1 + 2. | 3 | $NC + ND = NA + NB.$ Ax. 11. |
| Again | 4 | $NC + ND \sqsubset CD.$ 20. 1. |
| i. e. 3. | 5 | $NA + NB \sqsubset CD.$ Ax. 3. |
| i. e. | 6 | $AB \sqsubset CD.$ Q. E. D. |



E c

That

- 2 That which is nearest the Center, is greater than that which is further off.

P R E P A R A T I O N .

Draw NE, NF.

I say, for Example, $\angle CDE < \angle F$.



D E M O N S T R A T I O N .

- 1 $NC = NE, ND = NF.$ Def. 16. 1.
 And 2 $\angle CND < \angle ENF.$ Ax. 17.
 3 $NC = NE, ND = NF, \angle CND < \angle ENF.$
 4 $\angle CDE < \angle F.$ 24. 1. Q. E. D.

P R O P . XVI.

A Perpendicular Line CD, drawn thro' the Extremity of the Diameter of a Circle ALHG, is wholly without the Circle; and any other Right-line AL, drawn from the Point of Contact next the Circumference, falls within the Circle.

P R E P A R A T I O N .

1 Draw BF.

I say, $\angle CDE$ is wholly without the Circle.

DEMON-

DEMONSTRATION.



- 1 FAB is a $\text{r}\angle$. Def. 10. 1.
 ∴ e. 1. 2 $\angle FAB \sqcap \angle AFB$. 32. 1.
 ∴ e. 2. 3 $BF \sqsubset BA$. 19. 1.
 ∴ e. 3. 4 The Point F is further from the Center than the Point A.
 Now 5 The Point A is in the Circumference.
 ∴ 6 The Point F is without the Circumference.
 And 7 Since for the same Reason every other Point in CD is without the Circumference, besides A.
 ∴ 8 ~~CD~~ is wholly without the Circle. Q. E. D.

PREPARATION.

- 2 Draw $BE \perp AL$.

I say, AL falls within the Circle.

DEMONSTRATION.

- 1 BEA is a $\text{r}\angle$. Def. 10. 1.
 ∴ e. 1. 2 $\angle BEA \sqcap \angle BAE$. 32. 1.
 ∴ e. 2. 3 $BA \sqsubset BE$. 19. 1.
 ∴ e. 3. 4 The Point E is nearer the Center than the Point A.
 Now 5 The Point A is in the Circumference.
 ∴ 6 The Point E is within the Circumference.
 ∴ 7 AL falls within the Circle. Q. E. D.

COROLLARY.

A Right-line drawn thro' the Extremity of the Diameter of a Circle, and at Right-angles, is (Def. 2.) a Tangent to the said Circle.

E. e. 2.

PROP.

PROP. XVII.

From a Point given *A*, to draw a Right-line *AC*, which shall touch a Circle given *DCBF*.

PREPARATION.

From the Center *D*, of the Circle *DCBF*, let a Line *DA*, cutting the Circumference in *B*, be drawn to the given Point *A*; from the Center *D*, at the Distance *DA*, describe another Circle *DAEG*, and from *B*, draw a Perpendicular to *AD*, which will meet the Circle *DAEG*, in the Point *E*; then having drawn *ED*, meeting with the Circle *DCBF*, in the Point *C*, draw *AC*.

I say, *AC* touches the Circle *DCBF*.

DEMONSTRATION.



- | | | |
|--------------|--|-----------------|
| 1 | $DB = DC, DE = DA.$ | Def. 16. 1. |
| And 2 | $\angle D$ is common. | |
| ∴ e. 1. 2. 3 | $DB = DC, DE = DA, \angle D$ is common. | |
| ∴ e. 3. 4 | $\angle DBE = \angle DCA.$ | 4. 1. |
| But 5 | $\angle DBE$ is a $r\angle.$ | Def. 10. 1. |
| ∴ 6 | $\angle DCA$ is a $r\angle.$ | |
| ∴ e. 6. 7 | <i>AC</i> touches the Circle <i>DCBF</i> . | Cor. 16. Q.E.D. |

PROP. XVIII.

If a Right-line *AB*, touch a Circle *FEDC*, and from the Center *F*, to the Point of Contact *E*, a Right-line *FE* be drawn; that Line *FE*, is perpendicular to the Tangent *AB*.

PRE.

PREPARATION.

If FE be said not to be $\perp AB$, let any other Line FG be $\perp AB$, so ($\angle FGE$ being (Def. 10. 1.) Right, and consequently (32. 1.) $\sphericalangle FEG$) will FE be (19. 1.) $\sphericalangle$ that Line FG.



I say, $FE \perp AB$.

DEMONSTRATION.

- For 1 If any other Line, besides FE, should be $\perp AB$; as FG.
 Then 2 FE would be $\sphericalangle$ that Line FG.
 $\therefore$ e. 2. 3 AB would cut the Circumference.
 But 4 AB does not cut the Circumference. Def. 2.
 $\therefore$ 5 No other Line, besides FE, is $\perp AB$.
 $\therefore$ 6 $FE \perp AB$. Q. E. D.

PROP. XIX.

If a Right-line AB, touch a Circle CDEG, and from the Point of Contact C, a Right-line CE, be erected at Right-angles to the Tangent, the Center of the Circle is in the Right-line CE so erected.

PREPARATION.

If the Center be said not to be in CE, let it be in any Line CF, on either Side CE, so (CF being (18.) $\perp AB$, and consequently (Def. 10. 1.) $\angle FCB$ Right) will $\angle FCB$, form'd by that Line and the Tangent CB on either Side CE; be (Ax. 21.) $= \angle ECB$.



I say, The Center is in CE.

F f

DE.

DEMONSTRATION.

- For 1 If the Center should be in any Line.
 CF, on either Side CE.
 Then 2 $\angle FCB$, formed by that Line and the
 Tangent CB, would be $= \angle ECB$.
 But 3 $\angle FCB$ is either $\square$ or $\neg \angle ECB$.
 Ax. 17.
 $\therefore$ 4 The Center is not in any Line CF, on either Side CE.
 $\therefore$ 5 The Center is in CE. Q. E. D.



PROP. XX.

An Angle at the Center of a Circle ABDE, is always double to the Angle at the Circumference, when both the Angles stand upon the same Arch.

- Case 1 If one of the Sides DA, of the Angle at the Circumference D, be in a Right-line with one of the Sides CA, of the Angle at the Center BCA.

I say, $\angle BCA = 2 \angle D$.

DEMONSTRATION.

- 1 $CB = CD$. Def. 16. 1.
 $\therefore$ e. 1. 2 $\angle B = \angle D$. 5. 1.
 Again 3 $\angle BCA = \angle B + \angle D$. 32. 1.
 $\therefore$ e. 2. 3. 4 $\angle BCA = \angle D + \angle D$. Sub.
 i. e. 5 $\angle BCA = 2 \angle D$. Q. E. D.



- Case 2 If the two Sides DB, DF, of the Angle at the Circumference BDF, inclose the Angle at the Center BCF.

PRE-

PREPARATION.

Draw the Diameter DA.

I say, $\angle BCF = 2 \angle BDF$.

DEMONSTRATION.

- 1 $\angle BCA = 2 \angle BDA$. *Case 1.*
 And 2 $\angle FCA = 2 \angle FDA$. *Case 1.*
 1 + 2 3 $\angle BCA + \angle FCA = 2 \angle BDA + 2 \angle FDA$. *Ax. 11.*
 Also 4 $\angle BCA + \angle FCA = \angle BCF$. *Ax. 18.*
 ∴ c. 4. 3. 5 $\angle BCF = 2 \angle BDA + 2 \angle FDA$. *Ax. 1.*
 Again 6 $\angle BDF = \angle BDA + \angle FDA$. *Ax. 18.*
 6 × 2 7 $2 \angle BDF = 2 \angle BDA + 2 \angle FDA$. *Ax. 13.*
 ∴ c. 5. 7. 8 $\angle BCF = 2 \angle BDF$. *Ax. 1. Q. E. D.*



Case 3 If one of the Sides DB, of the Angle at the Circumference FDB, cut one of the Sides CF, of the Angle at the Center FCB.

PREPARATION.

From the angular Point D, draw the Diameter DA.

I say, $\angle FCB = 2 \angle FDB$.

DEMONSTRATION.

- 1 $\angle FCA = 2 \angle FDA$. *Case 1.*
 And 2 $\angle BCA = 2 \angle BDA$. *Case 1.*
 1 - 2 3 $\angle FCA - \angle BCA = 2 \angle FDA - 2 \angle BDA$. *Ax. 12.*
 Also 4 $\angle FCA - \angle BCA = \angle FCB$.
 ∴ c. 4. 3. 5 $\angle FCB = 2 \angle FDA - 2 \angle BDA$. *Ax. 1.*
 Again 6 $\angle FDB = \angle FDA - \angle BDA$.
 6 × 2 7 $2 \angle FDB = 2 \angle FDA - 2 \angle BDA$. *Ax. 13.*
 ∴ c. 5. 7. 8 $\angle FCB = 2 \angle FDB$. *Ax. 1. Q. E. D.*



F f 2

PROP.

PROP. XXI.

In a Circle ABCD, the Angles DAC, DBC, which are in the same Segment, are equal.

Case 1

If the Segment ABCD be greater than a Semicircle.

PREPARATION.

From the Center E, draw EC, ED.

I say, $\angle DAC = \angle DBC$.

DEMONSTRATION.



- 1 $\angle E = 2 \angle DAC$. 20.
 Also 2 $\angle E = 2 \angle DBC$. 20.
 $\therefore$ c. 1. 2. 3 $2 \angle DAC = 2 \angle DBC$. Ax. 1.
 $3 \div 2$ 4 $\angle DAC = \angle DBC$. Ax. 14. Q. E. D.

Case 2

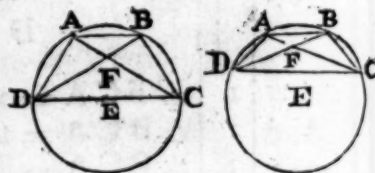
If the Segment ABCD, be equal to, or less than a Semicircle.

PREPARATION.

Draw AB.

I say, $\angle DAC = \angle DBC$.

DEMONSTRATION.



- 1 $\angle ADB = \angle BCA$. Case 1.
 i. e. 2 $\angle ADF = \angle BCF$.
 Again 3 $\angle DFA = \angle CFB$. 15. 1.
 And 4 $\angle DFA + \angle ADF + \angle DAF = \angle CFB + \angle BCF + \angle CBF$. Cor. 1. 32. 1.
 $3 + 2$ 5 $\angle DFA + \angle ADF = \angle CFB + \angle BCF$. Ax. 11.
 $4 - 5$ 6 $\angle DAF = \angle CBF$. Ax. 12.
 i. e. 7 $\angle DAC = \angle DBC$. Q. E. D.

PROP.

PROP. XXII.

The two opposite Angles of a quadrilateral Figure inscribed in a Circle, are taken together, equal to two Right-Angles.

PREPARATION.

Draw AC, BD.

I say, $\angle ABC + \angle ADC$, or, $\angle BCD + \angle BAD = 2r\angle$.

DEMONSTRATION.



- | | | | |
|-------------|---|--|----------|
| | 1 | $\angle BDA = \angle BCA$. | 21. |
| And | 2 | $\angle BDC = \angle BAC$. | 21. |
| 7 + 2 | 3 | $\angle BDA + \angle BDC = \angle BCA + \angle BAC$. | Ax. 11. |
| i. e. | 4 | $\angle ADC = \angle BCA + \angle BAC$. | |
| Again | 5 | $\angle ABC + \angle BCA + \angle BAC = 2r\angle$. | 32. 1. |
| v. c. 4. 5. | 6 | $\angle ABC + \angle ADC = 2r\angle$. | Sub. |
| Lastly | 7 | $\angle ABC + \angle ADC + \angle BCD + \angle BAD = 4r\angle$. | |
| | | Cor. 2. The. 1. 1. | |
| 7 - 6 | 8 | $\angle BCD + \angle BAD = 2r\angle$. | Ax. 12 |
| v. c. 6. 8. | 9 | $\angle ABC + \angle ADC$, or, $\angle BCD + \angle BAD = 2r\angle$. | Q. E. D. |

PROP. XXIII.

Two Similar and unequal Segments of Circles, ABC, ADC, cannot be set on the same Right-line AC, and the same Side thereof.



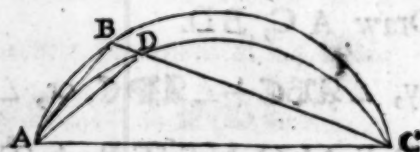
PRE.

PREPARATION.

Draw CB, cutting the Circumferences in D and B, and join AB, AD.

I say, $\angle ABC$, $\angle ADC$, are not similar.

DEMONSTRATION.



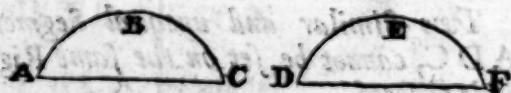
- For 1 If $\angle ABC$, $\angle ADC$ were similar.
 Then 2 $\angle ADC$ would be $= \angle ABC$. Def. 9.
 But 3 $\angle ADC < \angle ABC$. 16. 1.
 $\therefore$ 4 $\angle ABC$, $\angle ADC$, are not similar. Q. E. D.

PROP. XXIV.

Similar Segments of Circles, $\angle ABC$, $\angle DEF$, set upon equal Right-lines AC , DF , are equal one to the other.

That is, $\angle ABC = \angle DEF$.

DEMONSTRATION.



- For 1 If $\angle ABC$ be placed on $\angle DEF$.
 Then 2 AC being $= DF$. Hyp.
 $\therefore$ e. 2. 3 The Point A will fall on D, the Point C on F.
 $\therefore$ e. 3. 4 $\angle ABC$, $\angle DEF$, will stand on the same Right-line, and the same side thereof.
 $\therefore$ e. 4. 5 $\angle ABC = \angle DEF$. 23. Q. E. D.

PROP.

PROP. XXV.

The Segment ABC of a Circle being given, to find the Center of that Circle.

PREPARATION.

Choose at Pleasure three Points A, B, C, upon the Circumference ABC, and having joined the Right-lines BA, BC, and bisected them at the Points D, E, erect on those Points the Perpendiculars DG, EH.

I say, F is the Center.

DEMONSTRATION.



- | | | |
|----------------------------|----------------------------------|---|
| <p>1
2
3
4</p> | <p>c. 1.
c. 2.
c. 3.</p> | <p>DG bisects BA, EH bisects BC at $r\angle$. <i>Const.</i>
 The Center is in DG and EH. <i>Cor. 1.</i>
 The Center is in their common Intersection.
 F is the Center. Q. E. F. & D.</p> |
|----------------------------|----------------------------------|---|

Gg 2

PROP.

P R O P. XXVI.

In equal Circles, AICB, DLFE, equal Angles stand upon equal Arches of the Circumferences, AIC, DLF, whether those equal Angles be at the Centers, as G, H, or at the Circumferences, as B, E.

P R E P A R A T I O N.

Join AC, DF.

I say, Arch AIC = Arch DLF.

D E M O N S T R A T I O N.



- | | | | |
|------------------------|----|--|------------------|
| | 1 | $GA = HD, GC = HF.$ | Def. 1. |
| And | 2 | $\angle G = \angle H.$ | Hyp. |
| $\therefore$ e. 1. 2. | 3 | $GA = HD, GC = HF, \angle G = \angle H.$ | |
| $\therefore$ e. 3. | 4 | $AC = DF.$ | 4. 1. |
| $2 \div 2$ | 5 | $\frac{1}{2} \angle G = \frac{1}{2} \angle H.$ | Ax. 14. |
| Also | 6 | $\frac{1}{2} \angle G = \angle B.$ | 20. |
| $\therefore$ e. 5. 6. | 7 | $\frac{1}{2} \angle H = \angle E.$ | Ax. 8. |
| Also | 8 | $\frac{1}{2} \angle H = \angle E.$ | 20. |
| $\therefore$ e. 7. 8. | 9 | $\angle B = \angle E.$ | Ax. 8. |
| $\therefore$ e. 9. | 10 | The Segments ABC, DEF, are similar. | Def. 9. |
| $\therefore$ e. 10. 4. | 11 | The Segments ABC, DEF, are similar, and stand on equal Lines AC, DF. | |
| $\therefore$ e. 11. | 12 | Segment ABC = Segment DEF. | 24. |
| $\therefore$ e. 12. | 13 | Arch ABC = Arch DEF. | |
| Also | 14 | Arch ABC + Arch AIC = Arch DEF + Arch DLF. | Hyp. |
| 14 - 13. | 15 | Arch AIC = Arch DLF. | Ax. 12. Q. E. D. |

P R O P.

PROP. XXVII.

In equal Circles, ABC, DEF, the Angles standing upon equal Arches of the Circumferences AC, DF, are equal between themselves, whether they be made at the Centers, as G, H, or at the Circumferences, as B, E.

That is, $\angle G = \angle H$. $\angle B = \angle E$.

DEMONSTRATION.



- 1 AC, DF, are equal Arches of equal Circles. Hyp.
 2 The $\angle$ ' which they measure are equal.
 3 $\angle G = \angle H$.
 4 $\frac{1}{2} \angle G = \frac{1}{2} \angle H$. Ax. 14.
 5 $\angle E = \frac{1}{2} \angle H$. 20.
 6 $\frac{1}{2} \angle G = \angle E$. Ax. 8.
 7 $\frac{1}{2} \angle G = \angle B$. 20.
 8 $\angle B = \angle E$. Ax. 8.
 9 $\angle G = \angle H$. $\angle B = \angle E$. Q. E. D.

PROP. XXVIII.

In equal Circles, AICB, DKFE, equal Right-lines AC, DF, cut off equal Arches of the Circumferences, the greatest ABC equal to the greatest DEF, and the least AIC, equal to the least DKF.



H h

PRE.

PREPARATION.

From the Centers G, H , draw GA, GC, HD, HF .

I say, Arch $ABC = \text{Arch } DEF$. Arch $AIC = \text{Arch } DKF$.

DEMONSTRATION.



- | | | |
|-----------------------|---|--|
| | 1 | $GA = HD, GC = HF$. Def. 1. |
| And | 2 | $AC = DF$. Hyp. |
| $\therefore$ e. 1, 2. | 3 | $GA = HD, GC = HF, AC = DF$. |
| $\therefore$ e. 3. | 4 | $\angle G = \angle H$. 8. 1. |
| $\therefore$ e. 4. | 5 | Arch $AIC = \text{Arch } DKF$. 26. |
| Also | 6 | Arch $ABC + \text{Arch } AIC = \text{Arch } DEF + \text{Arch } DKF$. Hyp. |
| 6 — 5 | 7 | Arch $ABC = \text{Arch } DEF$. Ax. 12. |
| $\therefore$ e. 7. 5. | 8 | Arch $ABC = \text{Arch } DEF$. Arch $AIC = \text{Arch } DKF$. Q. E. D. |

PROP. XXIX.

In equal Circles, $AICB, DKFE$, equal Arches AIC, DKF , are subtended by equal Right-lines AC, DF .

PREPARATION.

From the Centers G, H , draw GA, GC, HD, HF , and join AC, DF .

I say, $AC = DF$.

DEMON-

DEMONSTRATION.



- 1 $GA = HD, GC = HF.$ Def. 1.
 And 2 Arch AIC = Arch DKF. Hyp.
 $\therefore$ e. 2. 3 $\angle G = \angle H.$ 27.
 $\therefore$ e. 1. 3. 4 $GA = HD, GC = HF, \angle G = \angle H.$
 $\therefore$ e. 4. 5 $AC = DF.$ 4. 1. Q. E. D.

Note. This and the three precedent Propositions may be understood also of the same Circle.

PROP. XXX.

To cut a Periphery given ABC into two equal Parts.

PREPARATION.

Draw the Right-line AC, and bisect it in D; from D draw a Perpendicular DB, meeting with the Arch in B, and join AB, CB.

I say, Arch AB = Arch CB.

DEMONSTRATION.



- 1 ADB, CDB. are $\text{r}\angle$'s. Def. 10. 1.
 $\therefore$ e. 1. 2 $\angle ADB = \angle CDB.$ Ax. 21.
 Again 3 $AD = DC, DB$ is common. Const.
 $\therefore$ e. 3. 2. 4 $AD = DC, DB$ is common, $\angle ADB = \angle CDB.$
 $\therefore$ e. 4. 5 $AB = CB.$ 4. 1.
 $\therefore$ e. 5. 6 Arch AB = Arch CB. 23. Q. E. F. & D.

H h 2

PROP.

P R O P. XXXI.

In a Circle ABFCG, the Angle ABC, which is in the Semicircle ABFC, is a Right-angle; the Angle BAC, which is in the greater Segment BAGC, is Acute, or less than a Right-angle; and the Angle BFC, which is in the lesser Segment BFC, is Obtuse, or greater than a Right-angle.

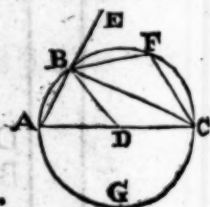
P R E P A R A T I O N.

From the Center D, draw DB, and produce AB to E.

I say, $\angle ABC$ is Right. $\angle BAC$ Acute. $\angle BFC$ Obtuse.

D E M O N S T R A T I O N.

- | | | | |
|----------------------------|----|---|--|
| | 1 | DA = DB. Def. 16. 1. | |
| $\therefore$ e. 1. | 2 | $\angle A = \angle DBA$. 5. 1. | |
| Again | 3 | DC = DB. Def. 16. 1. | |
| $\therefore$ e. 3. | 4 | $\angle DCB = \angle DBC$. 5. 1. | |
| 2 + 4. | 5 | $\angle A + \angle DCB = \angle DBA + \angle DBC$. Ax. 11. | |
| Also | 6 | $\angle A + \angle DCB = \angle EBC$. 32. 1. | |
| $\therefore$ e. 5. 6. | 7 | $\angle DBA + \angle DBC = \angle EBC$. Ax. 1. | |
| i. e. | 8 | $\angle ABC = \angle EBC$. | |
| $\therefore$ e. 8. | 9 | ABC is a $\text{r}\angle$. Def. 10. 1. | |
| $\therefore$ e. 9. | 10 | $\angle BAC$ is Acute. Cor. 1. 17. 1. | |
| Lastly | 11 | $\angle BAC + \angle BFC = 2\text{r}\angle$. 22. | |
| But e. 10. | 12 | $\angle BAC$ is Acute, or $\lhd$ a $\text{r}\angle$. | |
| $\therefore$ e. 11. 12. | 13 | $\angle BFC$ is Obtuse, or $\lhd$ a $\text{r}\angle$. | |
| $\therefore$ e. 9. 10. 13. | 14 | $\angle ABC$ is Right. $\angle BAC$ Acute. $\angle BFC$ Obtuse. | |
- Q. E. D.



P R O P.

PROP. XXXII.

If a Right-line AB touch a Circle, and from the Point of Contact C, be drawn a Right-line CE, cutting the Circle, the Angles ECB, ECA, which it makes with the Tangent AB, are equal to those Angles EDC, EFC, which are made in the alternate Segments of the Circle.

PREPARATION.

Let CD, the side of the Angle EDC, be perpendicular to AB, (for it is to the same Purpose, since (27.) all the Angles which stand on the Arch EFC are equal) and taking any Point F, in the Arch EC, join DE, EF, FC.

I say, $\angle ECB = \angle EDC$. $\angle ECA = \angle EFC$.

DEMONSTRATION.



- | | | |
|------------------|----|---|
| | 1 | CD $\perp$ AB. <i>Const.</i> |
| ∴ e. 1. | 2 | The Center is in CD. 19. |
| ∴ e. 2. | 3 | CD is the Diameter. <i>Def.</i> 17. 1. |
| ∴ e. 3. | 4 | CFED is a Semicircle. |
| ∴ e. 4. | 5 | DEC is a $\angle$. 31. |
| ∴ e. 5. | 6 | $\angle EDC + \angle ECD = 1\angle$. <i>Cor.</i> 3. 32. 1. |
| Again | 7 | DCB is a $\angle$. <i>Def.</i> 10. 1. |
| ∴ e. 7. | 8 | $\angle ECB + \angle ECD = 1\angle$. |
| ∴ e. 8. 6. | 9 | $\angle ECB + \angle ECD = \angle EDC + \angle ECD$. <i>Ax.</i> 1. |
| 9- $\angle ECD$ | 10 | $\angle ECB = \angle EDC$. <i>Ax.</i> 12. |
| Lastly | 11 | $\angle EDC + \angle EFC = 2\angle$. 22. |
| ∴ e. 10. 11. | 12 | $\angle ECB + \angle EFC = 2\angle$. <i>Sub.</i> |
| Also | 13 | $\angle ECB + \angle ECA = 2\angle$. 13. 1. |
| ∴ e. 13. 12. | 14 | $\angle ECB + \angle ECA = \angle ECB + \angle EFC$. <i>Ax.</i> 1. |
| 14- $\angle ECB$ | 15 | $\angle ECA = \angle EFC$. <i>Ax.</i> 12. |
| ∴ e. 10. 15. | 16 | $\angle ECB = \angle EDC$. $\angle ECA = \angle EFC$. Q. E. D. |

P R O P. XXXIII.

Upon a Right-line AB, to describe a Segment of a Circle AIEB, which shall contain an Angle AIB, equal to a given Right-lined Angle C.

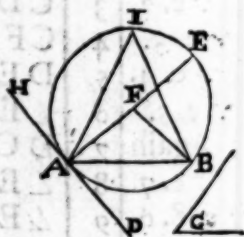
P R E P A R A T I O N.

Make (23. 1.) $\angle BAD = \angle C$. Through the Point A, draw (11. 1.) the Line AE perpendicular to HD. At one End of the Line AB, make $\angle ABF = \angle BAF$, and let one side thereof cut AE in F. From the Center F, thro' the Point A, describe a Circle, which (because $\angle ABF = \angle BAF$, and therefore (6. 1.) $FB = FA$) will also pass thro' B; and to any Point I, in the Periphery AIEB, draw AI, BI.

I say, $\angle AIB = \angle C$.

D E M O N S T R A T I O N.

- 1 HD is drawn through the Extremity of the Diameter AE at r^t $\angle$.
 2 $\therefore$ e. 1. HD touches the Circle. Cor. 16.
 Again 3 AB drawn from the Point of Contact, cuts the Circle.
 4 $\therefore$ e. 2. 3. HD touches, and AB drawn from the Point of Contact, cuts the Circle.
 5 $\therefore$ e. 4. $\angle AIB = \angle BAD$. 32.
 Also 6 $\angle C = \angle BAD$. Const.
 7 $\therefore$ e. 5. 6. $\angle AIB = \angle C$. Ax. 1. Q. E. F. & D.



P R O P. XXXIV.

From a Circle given ABC, to cut off a Segment ABC, containing an Angle B, equal to a Right-lined Angle given D.

P R E.

PREPARATION.

Draw (17.) a Right-line EF, touching the Circle given in A; also draw AC, making an $\angle FAC = \angle D$; and to any Point B, in the Periphery ABC, draw AB, CB.

I say, $\angle B = \angle D$.

DEMONSTRATION.

EF touches, and AC drawn from the point of Contact, cuts the Circle.

$\angle FAC = \angle B$. 32.

$\angle FAC = \angle D$. Const.

$\angle B = \angle D$. Ax. 1. Q. E. F. & D.



PROP. XXXV.

If in a Circle ACBD; two Right-lines AB, DC, cut each other, the Rectangle contained under the Segments AE, EB, of the one, is equal to the Rectangle contained under the Segments CE, ED, of the other.

Case 1. If the two Lines cut each other in the Center E.

I say, $AE \times EB = CE \times ED$.

DEMONSTRATION.

$AE = CE$. Def. 16. 1.

$EB = ED$. Def. 16. 1.

$AE \times EB = CE \times ED$. Ax. 13. Q. E. D.



Case 1 If one Line AB, pass through the Center F, and cut the other DC equally, and consequently (3.) at Right-Angles) in E.

PREPARATION.

Draw FD.

I say, $AE \times EB = CE \times ED$.

DEMONSTRATION.

- | | | |
|----------------|----------------------------------|-----------------|
| 1 | $AE \times EB + EFq = FBq$ | 9. 2. |
| And 2 | $FD = FB$ | Def. 16. r. |
| 2. 3 | $FDq = FBq$ | Ax. 9. |
| ∴ e. 1. 3. 4 | $AE \times EB + EFq = FDq$ | Am. r. |
| Also 5 | $EDq + EFq = FDq$ | 47. 1. |
| ∴ e. 4. 5. 6 | $AE \times EB + EFq = EDq + EFq$ | Ax. 1. |
| 6 - EFq. 7 | $AE \times EB = EDq$ | Ax. 12. |
| Again 8 | $CE = ED$ | Const. |
| And 9 | $CE \times ED = CE \times ED$ | |
| ∴ e. 8. 9. 10 | $CE \times ED = ED \times ED$ | Sub. |
| i.e. 11 | $CE \times ED = EDq$ | |
| ∴ e. 7. 11. 12 | $AE \times EB = CE \times ED$ | Ax. 1. Q. E. D. |

Case 3 If one Line AB, pass through the Center F, and cut the other DC unequally in E.

PREPARATION.

Bisect (3.) DC, by FG a Perpendicular from the Center; and draw FC.

I say, $AE \times EB = CE \times ED$.



DEMONSTRATION.



- | | | | |
|------------------------|----|--|------------------|
| | 1 | $CE \times ED + EGq = CGq.$ | 5. 2. |
| $1 + FGq$ | 2 | $CE \times ED + EGq + FGq = CGq + FGq.$ | Ax. 11. |
| Again | 3 | $EGq + FGq = FEq.$ | 47. 1. |
| $\therefore c. 2. 3.$ | 4 | $CE \times ED + FEq = CGq + FGq.$ | Sub. |
| Also | 5 | $FCq = CGq + FGq.$ | 47. 1. |
| $\therefore c. 4. 5.$ | 6 | $CE \times ED + FEq = FCq.$ | Ax. 1. |
| Lastly | 7 | $FB = FC.$ | Def. 16. 1. |
| $7 \text{ @ } 2.$ | 8 | $FBq = FCq.$ | Ax. 9. |
| $\therefore c. 6. 8.$ | 9 | $CE \times ED + FEq = FBq.$ | Ax. 1. |
| Also | 10 | $AE \times EB + FEq = FBq.$ | 5. 2. |
| $\therefore c. 10. 9.$ | 11 | $AE \times EB + FEq = CE \times ED + FEq.$ | Ax. 1. |
| $11 - FEq.$ | 12 | $AE \times EB = CE \times ED.$ | Ax. 12. Q. E. D. |

Case. 4 If neither of the two Lines AB, DC, pass thro' the Center F.

PREPARATION.

Thro' the Point of Intersection E, draw the Diameter GH.

I say, $AE \times EB = CE \times ED.$

DEMONSTRATION.



- | | | | |
|-----------------------|---|--------------------------------|-----------------|
| | 1 | $AE \times EB = GE \times EH.$ | Case 3. |
| Also | 2 | $CE \times ED = GE \times EH.$ | Case 3. |
| $\therefore c. 1. 2.$ | 3 | $AE \times EB = CE \times ED.$ | Ax. 1. Q. E. D. |

PROP. XXXVI.

A Tangent DB, and Secant DA, being drawn from the same Point D, taken at Pleasure without the Circle ABC, the Square of the Tangent DB, will be equal to the Rectangle under the whole Secant DA, and its external Part DC.

Case 1 If the Secant DA pass thro' the Center E.

PREPARATION.

From the Center E, to the Point of Contact B, draw EB, which (18.) will be Perpendicular to the Tangent DB.

I say, $DB^2 = DA \times DC$.

DEMONSTRATION.

1. $EB = EC$. Def. 16. 1.
 2. $EB^2 = EC^2$. Ax. 9.
 Again 3. $DB^2 + EB^2 = ED^2$. 47. 1.
 4. $DB^2 + EC^2 = ED^2$. Sub.
 Also 5. $DA \times DC + EC^2 = ED^2$. 6. 3.
 6. $DB^2 + EC^2 = DA \times DC + EC^2$. Ax. 1.
 6-EC². 7. $DB^2 = DA \times DC$. Ax. 13. Q. E. D.



Case 2 If the Secant DA pass not thro' the Center E.

PREPARATION.

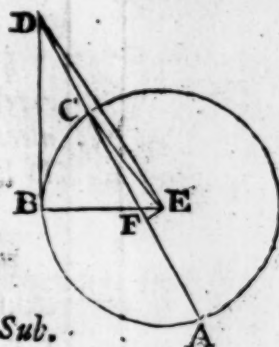
From the Center E, to the Point of Contact B, draw EB, which (18.) will be perpendicular to the Tangent DB; also draw EC, ED, and EF cutting DA, at Right-angles, which (3.) will Bisect CA in the Point F.

I say, $DB^2 = DA \times DC$.

DEMON-

DEMONSTRATION.

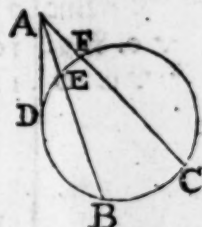
- 1 $DBq + EBq = EDq.$ 47. 1.
 Also 2 $DFq + EFq = EDq.$ 47. 1.
 $\therefore$ e. 1. 2. 3 $DBq + EBq = DFq + EFq.$ Ax. 1.
 Again 4 $DA \times DC + CFq = DFq.$ 6. 2.
 $\therefore$ e. 3. 4. 5 $DBq + EBq = DA \times DC + CFq + EFq.$ Sub.
 And 6 $CFq + EFq = ECq.$ 47. 1.
 $\therefore$ e. 5. 6. 7 $DBq + EBq = DA \times DC + ECq.$ Sub.
 Lastly 8 $EB = EC.$ Def. 16. 1.
 8 & 2. 9 $EBq = ECq.$ Ax. 9.
 $\therefore$ e. 7. 9. 10 $DBq + ECq = DA \times DC + ECq.$ Sub.
 10 $- ECq.$ 11 $DBq = DA \times DC.$ Ax. 12. Q. E. D.



COROLLARIES.

1 If from any Point A, taken without a Circle, there be drawn several Right-lines AB, AC, cutting the Circle, the Rectangles contained under the whole Lines AB, AC, and the external Parts AE, AF, are equal between themselves.

- For 1 If the Tangent AD be drawn.
 Then 2 $AB \times AE = ADq.$ 36.
 Also 3 $AC \times AF = ADq.$ 36.
 $\therefore$ e. 2. 3. 4 $AB \times AE = AC \times AF.$ Ax. 1.



2 If two Lines AB, AC, drawn from the same Point A, touch the Circle, those two Lines are equal one to the other.

- For 1 If the Secant AE be drawn.
 Then 2 $AE \times AF = ABq.$ 36.
 Also 3 $AE \times AF = ACq.$ 36.
 $\therefore$ e. 2. 3. 4 $ABq = ACq.$ Ax. 1.
 4 & 2. 5 $AB = AC.$ Ax. 10.



Kk 2

PROP.

PROP. XXXVII.

If the Rectangle contained under the Secant DA, and its external Part DC, be equal to the Square of a Line DB, meeting the Circumference of a Circle ABCF, that Line DB touches the Circle.

PREPARATION.

From the Point D, draw (17.) a Tangent DF; also from the Center B, draw EB, ED, EF.

I say, DB touches the Circle.

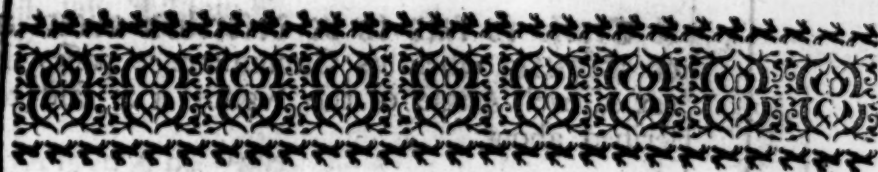
DEMONSTRATION.

- | | | |
|----------------------------|-----------------------------------|-------------------|
| 1 | $DA \times DC = DFq.$ | 36. |
| Allo 2 | $DA \times DC = DBq.$ | Hyp. |
| $\therefore$ c. 1. 2. 3 | $DFq = DBq.$ | Ax. 1. |
| 3 w 2. 4 | $DF = DB.$ | Ax. 10. |
| Again 5 | $EF = EB.$ | Def. 16. 1. |
| And 6 | ED is common. | |
| $\therefore$ c. 4. 5. 6. 7 | $DF = DB, EF = EB, ED$ is common. | |
| $\therefore$ c. 7. 8 | $\angle EFD = \angle EBD.$ | 8. 1. |
| But 9 | EFD is a r $\angle$. | 18. |
| $\therefore$ c. 8. 9. 10 | EBD is a r $\angle$. | |
| $\therefore$ c. 10. 11 | DB touches the Circle. | Cor. 16. Q. E. D. |



The End of the Third Book.





EUCLID'S ELEMENTS,

Demonstrated after a New, Plain, and Easy

METHOD.

BOOK IV.

DEFINITIONS.



Right-line is said to be applied in a Circle, when the Extreame thereof fall upon the Circumference.

As the Line AB .



A Right-lined figure is said to be inscribed in a Circle, when the Vertex of each of its Angles touches the Circumference of the Circle it is inscribed in.



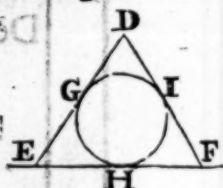
Thus the Triangle ABC is inscribed in the Circle ABC .

- 3 A Circle is said to be described about a Right-lined Figure, when its Circumference passes thro' the Vertex of each Angle of the Figure it is said to be circumscribed about.

Thus the Circle ABC is described about the Triangle ABC.

- 4 A Right-lined Figure is said to be described about a Circle, when all the Sides of the Figure which is circumscribed, touch the Periphery of the Circle.

Thus the Triangle EDF, is described about the Circle GHI.



- 5 A Circle is said to be inscribed in a Right-lined Figure, when the Periphery of the Circle touches all the Sides of the Figure in which it is inscribed.

Thus the Circle GHI, is inscribed in the Triangle EDF.

- 6 Regular Pentagons, Hexagons, and Pentedecagons, are such as have equal Sides, and equal Angles.

P R O P. I.

In a Circle given ABC, to apply a Right-line AB, equal to a Right-line given D, which doth not exceed AC the Diameter of the Circle.

P R E P A R A T I O N.

From the Center A, with the Distance $AE = D$, describe a Circle BEF, meeting with the Circle given in B, and draw AB.

I say, $\angle B = \angle D$.

DEMONSTRATION.



- 1 $AE = AB$. Def. 16. 1.
 Also 2 $AE = D$. Const.
 ∴ c. 1. 2. 3 $\angle B = \angle D$. Ax. 1. Q. E. F. & D.

PROP. II.

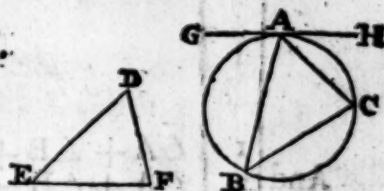
In a Circle given ABC, to inscribe a Triangle ABC, Equiangular to a Triangle given DEF.

PREPARATION.

Draw (17. 3.) the Tangent GH, and making (23. 1.) $\angle HAC = \angle E$, $\angle GAB = \angle F$, join BC; so (Def. 2.) is the Triangle ABC inscribed in the Circle ABC.

I say, $\angle B = \angle E$. $\angle C = \angle F$. $\angle CAB = \angle D$.

DEMONSTRATION.



- 1 $\angle HAC = \angle B$. 32. 3.
 Also 2 $\angle HAC = \angle E$. Const.
 ∴ c. 1. 2. 3 $\angle B = \angle E$. Ax. 1.
 Again 4 $\angle GAB = \angle C$. 32. 3.
 Also 5 $\angle GAB = \angle F$. Const.
 ∴ c. 4. 5. 6 $\angle C = \angle F$. Ax. 1.
 ∴ c. 3. 6. 7 $\angle B = \angle E$, $\angle C = \angle F$.
 ∴ c. 7. 8 $\angle CAB = \angle D$. Cor. 2. 32. 1.
 ∴ c. 7. 8. 9 $\angle B = \angle E$, $\angle C = \angle F$, $\angle CAB = \angle D$. Q. E. F. & D.

PROP. III.

About a Circle given ABC , to describe a Triangle KNM , Equiangular to a Triangle given DEF .

PREPARATION.

Produce the Side EF on both Sides; at the Center I , make (23. 1.) $\angle AIB = \angle DEG$, $\angle BIC = \angle DFH$. Thro' the Points A, B, C , draw (17. 3.) three Tangents meeting in K, N, M , so (Def. 4.) is the Triangle KNM , described about the Circle ABC ; also the Radij IA, IB, IC , are (18. 3.) perpendicular to KN, KM, MN , and consequently the Angles at A, B, C , are (Def. 10. 1.) Right.

I say, $\angle K = \angle DEF$. $\angle M = \angle DFE$. $\angle N = \angle EDG$.

DEMONSTRATION.



- | | |
|---------------------|---|
| 1 | $\angle A + \angle B + \angle AIB + \angle K = 4r\angle^\circ$. Cor. 2. The. 1. 1. |
| And 2 | $\angle A + \angle B = 2r\angle^\circ$. Const. |
| 1-2 3 | $\angle AIB + \angle K = 2r\angle^\circ$. Ax. 12. |
| Also 4 | $\angle DEG + \angle DEF = 2r\angle^\circ$. 13. 1. |
| e. 3. 4. 5 | $\angle AIB + \angle K = \angle DEG + \angle DEF$. Ax. 1. |
| But 6 | $\angle AIB = \angle DEG$. Const. |
| e. 5. 6. 7 | $\angle AIB + \angle K = \angle AIB + \angle DEF$. Sub. |
| 7- $\angle AIB$. 8 | $\angle K = \angle DEF$. Ax. 12. |
| Again 9 | $\angle B + \angle C + \angle BIC + \angle M = 4r\angle^\circ$. Cor. 2. The. 1. 1. |
| And 10 | $\angle B + \angle C = 2r\angle^\circ$. Const. |

$\angle BIC$

- 9-10. 11 $\angle BIC + \angle M = 2r\angle^\circ$. Ax. 12.
 Also 12 $\angle DFH + \angle DFE = 2r\angle^\circ$. 13. 1.
 ∴ e. 11. 12. 13 $\angle BIC + \angle M = \angle DFH + \angle DFE$. Ax. 1.
 Lastly 14 $\angle BIC = \angle DFH$. Conf.
 ∴ e. 13. 14. 15 $\angle BIC + \angle M = \angle BIC + \angle DFE$. Sub.
 15- $\angle BIC$. 16 $\angle M = \angle DFE$. Ax. 12.
 ∴ e. 8. 16. 17 $\angle K = \angle DEF$, $\angle M = \angle DFE$.
 ∴ e. 17. 18 $\angle N = \angle D$. Cor. 2. 32. 1.
 ∴ e. 17. 18. 19 $\angle B = \angle DCF$, $\angle M = \angle DFE$, $\angle N = \angle D$.
 Q. E. F. & D.

PROP. IV.

To inscribe a Circle EFG, in a given Triangle ABC.

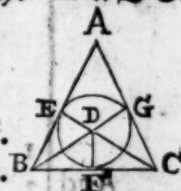
PREPARATION.

Bisect (9. 1.) the Angles B, C, with the Right-lines BD, CD, meeting in the Point D; then having (12. 1.) drawn the Perpendiculars DE, DF, DG, so that DEB, DFB, DFC, DGC, may (Def. 10. 1.) be Right-angles, from the Center D, with either of those Distances, describe the Circle EFG.

I say, The Circle EFG is inscribed in the $\triangle ABC$.

DEMONSTRATION.

- 1 $\angle DEB = \angle DFB$. Ax. 21.
 And 2 $\angle DBE = \angle DBF$, DB is common. Conf.
 ∴ e. 1. 2. 3 $\angle DEB = \angle DFB$, $\angle DBE = \angle DBF$, DB is com.
 ∴ e. 3. 4 $DE = DF$. 26. 1.
 Again 5 $\angle DFC = \angle DGC$. Ax. 21.
 And 6 $\angle DCF = \angle DCG$, DC is common. Conf.
 ∴ e. 5. 6. 7 $\angle DFC = \angle DGC$, $\angle DCF = \angle DCG$, DC is common.
 ∴ e. 7. 8 $DF = DG$. 26. 1.
 ∴ e. 4. 8. 9 $DE = DF = DG$. Ax. 1.
 ∴ e. 9. 10 The Circle EFG passes thro' the Points E, F, G. Def. 16. 1.
 Also 11 DEB, DFB, DGC, are $r\angle^\circ$. Conf.
 ∴ e. 11. 12 The Periphery of the Circle EFG touches all the Sides of the $\triangle ABC$. Cor. 16. 3.
 ∴ e. 12. 13 The Circle EFG is inscribed in the $\triangle ABC$. Def. 5.
 Q. E. F. & D. M m PROP.



PROP. V.

About a given Triangle ABC , to describe a Circle ABC .

PREPARATION.

Bisect (ro. 1) any two Sides AB , AC , in the Points D , E , and having (11. 1.) drawn from those Points D , E , the Lines DF , EF , at Right-angles to the Sides AB , AC , (which will meet, either within the Triangle ABC , or in the Line BC , or without in the Point F ;) draw FA , FB , FC , and from the Point F , with either of those Distances, describe the Circle ABC .

I say, The Circle ABC is described about the $\triangle ABC$.

DEMONSTRATION.



- | | | |
|-----------------------|----|---|
| | 1 | $AD = DB$, DF is common. <i>Const.</i> |
| And | 2 | $\angle FDA = \angle FDB$. <i>Ax. 21.</i> |
| $\therefore$ e. 1. 2. | 3 | $AD = DB$, DF is common, $\angle FDA = \angle FDB$. |
| $\therefore$ e. 3. | 4 | $FA = FB$. 4. 1. |
| Again | 5 | $AE = EC$, FE is common. <i>Const.</i> |
| And | 6 | $\angle FEA = \angle FEC$. <i>Ax. 21.</i> |
| $\therefore$ e. 5. 6. | 7 | $AE = EC$, FE is common, $\angle FEA = \angle FEC$. |
| $\therefore$ e. 7. | 8 | $FA = FC$. 4. 1. |
| $\therefore$ e. 4. 8. | 9 | $FA = FB = FC$. <i>Ax. 1.</i> |
| $\therefore$ e. 9. | 10 | The Circumference of the Circle ABC , passes thro' the Vertex of each of the $\angle$'s A , B , C . <i>Def. 16. 1.</i> |
| $\therefore$ e. 10. | 11 | The Circle ABC is described about the $\triangle ABC$. <i>Def. 3. Q. E. F. & D.</i> |

SCHC.

SCHOLIUM.

By the same Method, may a Circle be described, that shall pass thro' three Points given, not being in the same straight Line.

PROP. VI.

In a Circle given ABCD, to inscribe a Square ABCD.

PREPARATION.

Draw (11. 1.) the Diameters AC, DB, cutting each other at Right-angles in the Center E; then join their Extremes A, B, C, D, with the Right-lines AB, BC, CD, DA.

I say, The Square ABCD is inscribed in the Circle ABCD.

DEMONSTRATION.



- 1 $\angle AEB, BEC, CED, DEA$, are $r\angle$. *Const.*
- 2 $\angle AEB = \angle BEC = \angle CED = \angle DEA$. *Ax. 21.*
- 3 Arch AB = Arch BC = Arch CD = Arch DA. *26. 3.*
- 4 $AB = BC = CD = DA$. *29. 3.*
- 5 ABC, BCD, CDA, DAB, are Semicircles. *Def. 18. 1.*
- 6 ABC, BCD, CDA, DAB, are $r\angle$. *31. 3.*
- 7 $\angle ABC = \angle BCD = \angle CDA = \angle DAB$. *Ax. 21.*
- 8 $AB = BC = CD = DA; \angle ABC = \angle BCD = \angle CDA = \angle DAB$.
- 9 ABCD is a Square. *Def. 29. 1.*
- 10 The Vertex of each of its Angles touches the Circumference of the Circle ABCD.
- 11 The Square ABCD is inscribed in the Circle ABCD. *Def. 2. Q. E. F. & D.*

PROP. VII.

About a Circle given ABCD, to describe a Square FHIG.

PREPARATION.

Draw (11. 1.) the Diameters AC, DB, cutting each other at Right-angles in the Center E; then thro' their Extremes A, B, C, D, draw (17. 3.) Tangents meeting in F, H, I, G, so (18. 2.) will EA, EB, EC, ED be perpendicular to those Tangents, and therefore (Def. 10. 1.) will the Angles at A, B, C, D, be Right.

I say, The Square FHIG is described about the Circle ABCD.

DEMONSTRATION.

- 1 The $\angle$'s at A, B, C, D, E, are Right. *Const.*
- 2 $FG \parallel HI, FH \parallel GI, FA \parallel HC, EH \parallel AC, EB \parallel GD, EG \parallel BD.$
28. 1
- 3 FHIG, FHCA, FBDG, are $\square$'s. *Def. 35. 1.*
- 4 $FH = AC, BD = FG.$ 34. 1.
- Also 5 $AC = BD.$ *Def. 16. 1.*
- 6 $FH = AC, AC = FG.$ *Sub.*
- 7 $FH = FG.$ *Ax. 1.*
- Also e. 3. 8 $HI = FG, FH = GI.$ 34. 1.
- 9 $HI = FG = FH = GI.$ *Ax. 1.*
- Again 10 $\angle B + \angle E + \angle A + \angle F = 4r\angle.$ $\angle A + \angle E + \angle D + \angle G = 4r\angle.$ *Cor. 2. Theo. 1. 1.*
- Also 11 $\angle B + \angle E + \angle A = 3r\angle.$ $\angle A + \angle E + \angle D = 3r\angle.$ *Conj.*
- Also e. 10. 11. 12 $\angle F = r\angle, \angle G = r\angle.$
- Also e. 12. 13 $\angle F = \angle G.$ *Ax. 21.*
- Lastly 14 $\angle I = \angle F, \angle G = \angle H.$ 34. 1.
- Also e. 13. 14. 15 $\angle I = \angle F = \angle G = \angle H.$ *Ax. 1.*
- Also e. 9. 15. 16 $HI = FG = FH = GI; \angle I = \angle F = \angle G = \angle H.$
- Also e. 16. 17 FHIG is a Square. *Def. 29. 1.*
- Also 18 All its Sides touch the Periphery of the Circle ABCD. *Const.*
- Also e. 18. 19 The Square FHIG is described about the Circle ABCD. *Def. 4. Q. E. F. & D.*



PROP.

P R O P. VIII.

In a Square given ABCD, to inscribe a Circle HEFG.

P R E P A R A T I O N.

Bisect (10. 1.) the opposite Sides of the Square ABCD, in the Points E, G, H, F, which being (Def. 29. 1.) equal, as also (28. 1.) parallel, their Halves AE, DG, EB, GC, AH, BF, HD, FC, will (Ax. 8.) be equal, as also parallel; then having drawn EG, HF, from the Point of Intersection I, at the Distance IH, describe the Circle HEFG.

I say, The Circle HEFG is inscribed in the Square ABCD.

D E M O N S T R A T I O N.



- 1 AE || and = DG; EB || and = GC. *Const.*
- 2 AD || EG; EG || BC. 33. 1.
- 3 AH || and = BF; HD || and = FC. *Const.*
- 4 AB || HF; HF || DC. 33. 1.
- 5 AD || EG, EG || BC, AB || HF, HF || DC.
- 6 IA, IB, IC, are =. *Def. 35. 1.*
- 7 IH = AE, AH = IE, IE = BF, EB = IF, IF = GC, FC = IG. 34. 1.
- 8 AE = AH, BF = EB, GC = FC. *Const.*
- 9 IH = AE, AE = IE; IE = BF, BF = IF; IF = GC, GC = IG. *Sub.*
- 10 IH = IE = IF = IG. *Ax. 1.*
- 11 The Circle HEFG passes thro' the Points H, E, F, G. *Def. 16. 1.*
- 12 IH || AE, IE || AH, IF || GC, IG || FC. *Def. 35. 1.*
- 13 $\angle A + \angle H = 2r\angle$, $\angle A + \angle E = 2r\angle$, $\angle C + \angle F = 2r\angle$, $\angle C + \angle G = 2r\angle$. 29. 1.
- 14 $\angle A = r\angle$, $\angle C = r\angle$. *Def. 29. 1.*
- 15 H, E, F, G, are $r\angle$.
- 16 The Periphery of the Circle HEFG touches all the Sides of the Square ABCD. *Cor. 16. 3.*
- 17 The Circle HEFG is inscribed in the Square ABCD. *Def. 5. Q. E. F. & D.*

PROP. IX.

About a given Square $ADBC$, to describe a Circle $ADBC$.

PREPARATION.

Draw the Diagonals AB , CD , cutting each other in the Point E ; then from the Center E , at the Distance EA , describe the Circle $ADBC$.

I say, The Circle $ADBC$ is described about the Square $ADBC$.



DEMONSTRATION.

- 1 $CA = CB$, $\angle ACB$ is Right; $DA = DB$, $\angle ADB$ is Right. *Def. 29. 1.*
- 2 $\angle CAB$, DAB , DBA , are Semiright. *Cor. 6. 32. 1.*
- 3 $AC = AD$, $\angle DAC$ is Right; $BC = BD$, $\angle DBC$ is Right. *Def. 29. 1.*
- 4 $\angle ACD$, ADC , BDC , are Semiright. *Cor. 6. 32. 1.*
- 5 $\angle CAB = \angle ACD$, $\angle DAB = \angle ADC$, $\angle DBA = \angle BDC$.
- 6 $\angle CAE = \angle ACE$, $\angle DAE = \angle ADE$, $\angle DBE = \angle BDE$.
- 7 $EC = EA$, $EA = ED$, $ED = EB$. *6. 1.*
- 8 $EC = EA = ED = EB$. *Ax. 1.*
- 9 The Circumference of the Circle $ADBC$, passes thro' the Vertex of each of the $\angle A$, D , B , C . *Def. 16. 1.*
- 10 The Circle $ADBC$ is described about the Square $ADBC$. *Def. 3. Q. E. F. & D.*

PROP. X.

To make an Isosceles Triangle ABC , having each Angle at the Base, A , BCA , double to the remaining Angle B .

PREPARATION.

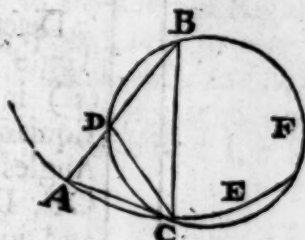
Draw the Line BA at pleasure, and (11. 2.) divide it at the Point D , so that the Square of BD , be equal to the Rectangle under BA , AD ; then from the Center B , at the Distance BA , describe the Arch ACE , in which

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PROP.

PROP. XI.

In a Circle given $ABCDE$, to inscribe a regular Pentagon $ABCDE$.

PREPARATION.

Describe (10.) an Isosceles Triangle FGH , having each Angle at the Base double to the remaining Angle, and (2.) inscribe in the given Circle, the Triangle ACD Equiangular to the Triangle FGH ; then (9. 1.) bisect the Angles at the Base ACD , ADC , with the Right-lines CE , DB , meeting with the Circumference in E and B , and join CB , BA , AE , ED .

I say, The regular Pentagon $ABCDE$ is inscribed in the Circle $ABCDE$.

DEMONSTRATION.



- | | | |
|-----------------------|----|---|
| | 1 | $\triangle ACD = 2 \angle CAD = \angle ADC.$ <i>Const.</i> |
| $\therefore$ e. 1. | 2 | $\angle ACD = \angle ADC.$ <i>Ax. 7.</i> |
| $\therefore$ e. 2. | 3 | $\angle ACE = \angle ECD = \angle ADB = \angle BDC.$ <i>Ax. 8.</i> |
| Again | 4 | $\angle BDC = \angle CAD.$ <i>Const.</i> |
| $\therefore$ e. 3. 4. | 5 | $\angle ACE = \angle ECD = \angle ADB = \angle BDC = \angle CAD.$ <i>Ax. 1.</i> |
| $\therefore$ e. 5. | 6 | Arch $AE =$ Arch $ED =$ Arch $AB =$ Arch $BC =$ Arch $CD.$ 26. 3. |
| $\therefore$ e. 6. | 7 | $AE = ED = AB = BC = CD.$ 29. 3. |
| Lastly | 8 | Each of the $\angle$'s ABC , BCD , CDE , DEA , EAB , stand on three of the equal Arches. |
| $\therefore$ e. 8. | 9 | $\angle ABC = \angle BCD = \angle CDE = \angle DEA = \angle EAB.$ 27. 3. |
| $\therefore$ e. 7. 9. | 10 | $AE = ED = AB = BC = CD$; $\angle ABC = \angle BCD = \angle CDE = \angle DEA = \angle EAB.$ |
| $\therefore$ e. 10. | 11 | $ABCDE$ is a regular Pentagon. <i>Def. 6.</i> |
| Also | 12 | The Vertex of each of its Angles touches the Circumference of the Circle $ABCDE$. |
| $\therefore$ e. 12. | 13 | The regular Pentagon $ABCDE$ is inscribed in the Circle $ABCDE.$ <i>Def. 2.</i> Q. E. F. & D. |

PROP.

PROP. XII.

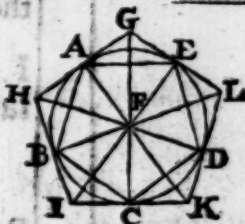
About a Circle given $ABCDE$, to describe a regular Pentagon $GHIKL$.

PREPARATION.

Inscribe (11.) a regular Pentagon $ABCDE$ in the Circle given; then having from the Center F , drawn the Right-lines FA, FB, FC, FD, FE , draw to those Lines so many Perpendiculars GAH, HBI, ICK, KDL, LEG , so (Def. 10. 1.) will all the Angles at A, B, C, D, E , be Right, and consequently (Ax. 21.) equal. Lastly, draw FG, FH, FI, FK, FL .

I say, The regular Pentagon $GHIKL$ is described about the Circle $ABCDE$.

DEMONSTRATION.



- 1 GA, GE , touch the Circle. Cor. 16. 3.
- 2 $GA = GE$. Cor. 2. 36. 3.
- 3 $FA = FE$. Def. 16. 1.
- 4 FG is common.
- 5 $GA = GE, FA = FE, FG$ is common.
- 6 $\angle GFA = \angle GFE$. 8. 1.
- 7 $\angle AFB$ is bisected.
- 8 After the same manner, are the $\angle AFB, BFC, CFD, DFE$, bisected.
- 9 $\angle AFE = \angle AFB = \angle BFC = \angle CFD = \angle DFE$. 27. 3.
- 10 All the $\angle$'s at F , their Halves, are equal. Ax. 8.
- 11 All the $\angle$'s, at A, B, C, D, E , are equal. Const.
- 12 $FA = FB = FC = FD = FE$. Def. 16. 1.
- 13 FA, FB, FC, FD, FE , all the $\angle$'s at A, B, C, D, E , as also all those at F , are equal.
- 14 $GA, AH, HB, BI, IC, CK, KD, DE, LE, EG$, are equal. 26. 1.
- 15 GH, HI, IK, KL, LG , their Doubles, are equal. Ax. 7.
- 16 The ten $\angle$'s at G, H, I, K, L , are equal. 26. 1.
- 17 The five $\angle$'s G, H, I, K, L , their Doubles, are equal. Ax. 7.
- 18 $GH = HI = IK = KL = LG, \angle G = \angle H = \angle I = \angle K = \angle L$.
- 19 $GHIKL$ is a regular Pentagon. Def. 6.
- 20 All its Sides touch the Periphery of the Circle $ABCDE$. Cor. 16. 3.
- 21 The regular Pentagon $GHIKL$ is described about the Circle $ABCDE$. Def. 4. Q. E. F. & D.

O O

PROP.

PROP. XIII.

In a regular Pentagon given $ABCDE$, to inscribe a Circle $LGH IK$.

PREPARATION.

Bisect (*9. 1.*) any two nearest Angles of the Pentagon A and B , with the Right-lines AF , BF , meeting in the Point F ; then having from F , drawn the Perpendiculars FL , FG , FH , FI , FK , so that the Angles at L , G , H , I , K , may (*Def. 10. 1.*) be Right, and consequently (*Ax. 21.*) equal, draw FC , FD , FE , and from the Center F , at the Distance FL , describe the Circle $LGH IK$.

I say, The Circle $LGH IK$ is inscribed in the regular Pentagon $ABCDE$.

DEMONSTRATION.



- | | |
|----------------------------|--|
| 1 | $BA = BC$. <i>Def. 6.</i> |
| And 2 | BF is common, $\angle FBA = \angle FBC$. <i>Const.</i> |
| <i>∴ e. 1. 2.</i> 3 | $BA = BC$, BF is common, $\angle FBA = \angle FBC$. |
| <i>∴ e. 3.</i> 4 | $\angle FAB = \angle FCB$. <i>4. 1.</i> |
| Again 5 | $\angle C = \angle A$. <i>Def. 6.</i> |
| <i>5 ÷ 2</i> 6 | $\div \angle C = \div \angle A$. <i>Ax. 14.</i> |
| Also 7 | $\angle FAB = \div \angle A$. <i>Const.</i> |
| <i>∴ e. 7. 6.</i> 8 | $\angle FAB = \div \angle C$. <i>Ax. 8.</i> |
| <i>∴ e. 4. 8.</i> 9 | $\angle FCB = \div \angle C$. <i>Ax. 14.</i> |
| <i>∴ e. 9.</i> 10 | $\angle C$ is bisected. |
| Also 11 | After the same manner is $\angle D$ bisected. |
| <i>∴</i> 12 | The four whole $\angle$'s at A, B, C, D , are bisected. |
| Now 13 | The four whole $\angle$'s at A, B, C, D , are equal. <i>Def. 6.</i> |
| <i>∴ e. 13.</i> 14 | The eight $\angle$'s at A, B, C, D , their Halves, are equal. <i>Ax. 8.</i> |
| Also 15 | All the $\angle$'s at L, G, H, I, K , are equal. <i>Const.</i> |
| And 16 | $FA, FG, FB, FH, FC, FI, FD$, are common. |
| <i>∴ e. 16. 15. 14.</i> 17 | $FA, FG, FB, FH, FC, FI, FD$, are common; all the $\angle$'s at L, G, H, I, K , as also all at A, B, C, D , are equal. |

The

- $\therefore$ c. 17. 18 The eight Δ 's from L to K, are equal. 26. 1.
 $\therefore$ c. 18. 19 $FL = FG = FH = FI = FK$. 26. 1.
 $\therefore$ c. 19. 20 The Circle LGHIK passes thro' the Points L, G, H, I, K. Def. 16. 1.
 Also. 21 The Periphery of the Circle LGHIK, touches all the Sides of the regular Pentagon. Cor. 16. 3.
 $\therefore$ c. 21. 22 The Circle LGHIK is inscribed in the regular Pentagon ABCDE. Def. 3. Q. E. F. & D.

COROLLARY.

If any two nearest Angles of an Equilateral and Equiangular Figure be bisected, and from that Point in which the Lines meet that bisect the Angles, be drawn Right-lines to the remaining Angles of the Figure, all the Angles of the Figure shall be bisected.

PROP. XIV.

About a regular Pentagon given ABCDE, to describe a Circle ABCDE.

PREPARATION.

Bisect (9. 1.) any two nearest Angles of the Pentagon, with the Right-lines AF, BF, meeting in the Point F; then from the Point F, draw the Right lines FC, FD, FE, so (Cor. 13.) will the Angles C, D, E, be also bisected.

I say, The Circle ABCDE, is described about the regular Pentagon ABCDE.

DEMONSTRATION.

- $\therefore$ c. 1. 1 $\angle A = \angle B, \angle B = \angle C$. Def. 6.
 $\therefore$ c. 2. 2 $\angle FAB = \angle FBA, \angle FBC = \angle FCB$. Ax. 8.
 $\therefore$ c. 3. 3 $FA = FB, FB = FC$. 6. 1.
 Again 4 $\angle C = \angle D, \angle D = \angle E$. Def. 6.
 $\therefore$ c. 4. 5 $\angle FCD = \angle FDC, \angle FDE = \angle FED$. Ax. 8.
 $\therefore$ c. 5. 6 $FC = FD, FD = FE$. 6. 1.
 $\therefore$ c. 3. 6. 7 $FA = FB = FC = FD = FE$. Ax. 1.
 $\therefore$ c. 7. 8 The Circumference of the Circle ABCDE passes thro' the Vertex of each of the $\angle$'s A, B, C, D, E. Def. 16. 1.
 $\therefore$ c. 8. 9 The Circle ABCDE, is described about the regular Pentagon ABCDE. Def. 3. Q. E. F. & D.



PROP. XV.

In a Circle given $ABCDEF$, to inscribe a regular Hexagon $ABCDEF$.

PREPARATION.

Draw the Diameter AD , and from its extremity D , thro' the Center G , describe the Arch CDE ; then from the Points C, E , draw the Diameters CF, EB , and join CD, DE, EF, FA, AB, BC .

I say, The regular Hexagon $ABCDEF$, is inscribed in the Circle $ABCDEF$.

DEMONSTRATION.



- 1 $DC = DG = GC$; $DE = DG = GE$. Def. 16. 1.
- 2 $\angle CGD = 60^\circ$, $\angle DGE = 60^\circ$. Cor. 4. 32. 1.
- 3 $\angle CGD = \angle FGA$; $\angle DGE = \angle AGB$. 15. 1.
- 4 $\angle FGA = 60^\circ$, $\angle AGB = 60^\circ$. Ax. 1.
- 5 $\angle FGA + \angle AGB = 120^\circ$.
- 6 $\angle FGA + \angle AGB + \angle BGC = 180^\circ$. Cor. 2. 13. 1.
- 7 $\angle BGC = 60^\circ$. Ax. 12.
- 8 $\angle BGC = \angle EGE$. 15. 1.
- 9 $\angle EGF = 60^\circ$. Ax. 1.
- 10 $\angle CGD = \angle DGE = \angle EGF = \angle FGA = \angle AGB = \angle BGC$. Ax. 1.
- 11 Arch $CD =$ Arch $DE =$ Arch $EF =$ Arch $FA =$ Arch $AB =$ Arch BC . 26. 3.
- 12 $CD = DE = EF = FA = AB = BC$. 29. 3.
- 13 Each of the $\angle CDE, DEF, EFA, FAB, ABC, BCD$, stand on four of the equal Arches.
- 14 $\angle CDE = \angle DEF = \angle EFA = \angle FAB = \angle ABC = \angle BCD$. 27. 3.
- 15 $CD = DE = EF = FA = AB = BC$; $\angle CDE = \angle DEF = \angle EFA = \angle FAB = \angle ABC = \angle BCD$.
- 16 $ABCDEF$ is a regular Hexagon. Def. 6.
- 17 The Vertex of each of its Angles touches the Circumference of the Circle $ABCDEF$.
- 18 The regular Hexagon $ABCDEF$, is inscribed in the Circle $ABCDEF$. Def. 2. Q. E. F. & D.

COROL.

COROLLARY.

The Side of a regular Hexagon inscribed in a Circle, is equal the Semidiameter.

PROP. XVI.

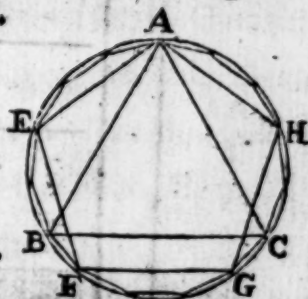
In a Circle given AEBC, to inscribe a regular Pentecagon.

PREPARATION.

Inscribe (2.) in the Circle AEBC, an Equilateral Triangle ABC, as also (11.) a regular Pentagon AEF GH, so that the Angles may meet at the Point A, and join BF.

I say, The Pentecagon whose Side is BF, is regular and inscribed in the Circle AEBC.

DEMONSTRATION.



1. Arch AF = $\frac{1}{5}$ of the Periphery. *Const.*
- i. e. 2. Arch AF = $\frac{1}{5}$ of the Periphery.
- Also 3. Arch AB = $\frac{1}{3}$ of the Periphery. *Const.*
- i. e. 4. Arch AB = $\frac{1}{3}$ of the Periphery.
- 2 — 4. 5. Arch AF — Arch AB = $\frac{1}{5} - \frac{1}{3}$ of the Periphery. *Ax. 12.*
- i. e. 6. Arch BF = $\frac{1}{5}$ of the Periphery.
- ∴ e. 6. 7. The Pentecagon whose Side is BF, is Equilateral.
- Again 8. Each of its Angles stand on Thirteen of the equal Arches.
- ∴ e. 8. 9. The Pentecagon whose Side is BF, is Equiangular. 27 3.
- ∴ e. 7. 9. 10. The Pentecagon whose Side is BF, has equal Sides and equal Angles.
- ∴ e. 10. 11. The Pentecagon whose Side is BF, is regular. *Def. 6.*
- Also 12. The Vertex of each of its Angles touches the Circumference of the Circle AEBC.
- ∴ e. 12. 13. The Pentecagon whose Side is BF, is regular and inscribed in the Circle AEBC. *Def. 2. Q. E. F. & D.*

The End of the Fourth Book.

P P

EUCLID'S

EUCLID's ELEMENTS,

Demonstrated after a New, Plain, and Easie
METHOD.

BOOK. V.

DEFINITIONS.



Ratio (or rate) is the mutual Habitude or Respect of two Quantities of the same Kind each to other, according to Quantity.

In every Ratio, that Quantity which is referred to another Quantity, is called the **Antecedent** of the Ratio, and that to which the other is referred, is called the **Consequent** of the Ratio; as in the Ratio of 6 to 3, 6 is the **Antecedent**, and 3 the **Consequent**.

The Quantity of any Ratio is known by dividing the Antecedent by the Consequent; as the Ratio of 6 to 3, is expressed by $\frac{6}{3} = 2$, or the Quantity of the Ratio of A to B, is $\frac{A}{B}$.

Where-

Wherefore, often for Brevities Sake, we denote the Quantities of Ratio's, thus; $\frac{A}{B}$ $\square$ or $=$ or $\sqsupset \frac{C}{D}$. that is, the Ratio of A to B is greater, equal, or less than the Ratio of C to D.

- 2 **Proportional Quantities**, whether **Continued**, or **Discontinued**, are such as have the same Ratio. Thus the **Continued** Quantities ARRR, ARR, AR, A, or 16, 8 4 2, are proportional, because the Ratio of AR³ to AR², or of 16 to 8, that is, $\frac{AR^3}{AR^2} = R$, or $\frac{16}{8} = 2$, is the same with that of AR to A, or 4 to 2, that is, with $\frac{AR}{A} = R$, or $\frac{4}{2} = 2$. Also the **Discontinued** Quantities AR, A, BR, B, or 4, 2, 6, 3, are proportional, because the Ratio, of AR to A, or of 4 to 2, that is, $\frac{AR}{A} = R$, or $\frac{4}{2} = 2$, is the same with that of BR to B, or of 6 to 3, that is, with $\frac{BR}{B} = R$, or $\frac{6}{3} = 2$.

Whence equal Ratio's may be expressed as are Proportionals, and Proportionals as are equal Ratio's.

Thus $\frac{AR}{A} = \frac{BR}{B}$, therefore AR:A::BR:B. also AR:A::

BR:B. therefore $\frac{AR}{A} = \frac{BR}{B}$

- 3 **Inverse Ratio**, is when the Consequent is taken as the Antecedent, and so compared to the Antecedent as the Consequent. As if A:C::B:C. therefore by **Inversion**, C:A::C:B.

4 **Alternate Proposition**, is the Comparing of Antecedent to Antecedent, and Consequent to Consequent.

As $ARRR : ARR :: AR : A$. or $AR : A :: BR : B$. therefore **Alternately**, or by **Permutation**, $AR^3 : AR :: AR : A$. or $AR : BR :: A : B$.

5 **Divided Ratio**, is when the Excess wherein the Antecedent exceeds the Consequent, is compared to the Consequent.

As $ARRR : ARR :: AR : A$. or $AR : A :: BR : B$. therefore by **Division**, $AR^3 - AR^2 : AR^2 :: AR - A : A$. or $AR - A : A :: BR - B : B$.

6 **Compound Ratio**, is when the Antecedent and Consequent taken both as one, are compared to the Consequent it self.

As $ARRR : ARR :: AR : A$. or $AR : A :: BR : B$. therefore by **Composition**, $AR^3 + AR^2 : AR^2 :: AR + A : A$. or $AR + A : A :: BR + B : B$.

7 **Converse Ratio**, is when the Antecedent is compared to the Excess wherein the Antecedent exceeds the Consequent.

As if $AB + BC : AB :: DE + EF : DE$. therefore by **Conversion**, $AB + BC : BC :: DE + EF : EF$.

Note: The Six first Propositions of this Book, serving only for the Proof of those that follow, by the Method of Equimultiples, my intent being not to use that Method, are here omitted.

PROP. VII.

Equal Quantities A, B, have to the same Quantity C, the same Proportion; and the same Quantity C, has the same Proportion to equal Quantities A, B.

That is, $A:C::B:C$. $C:A::C:B$.

DEMONSTRATION.

1 $A=B$. Hyp.
 And 2 $C=C$.
 1 ÷ 2. 3 $\frac{A}{C} = \frac{B}{C}$. Ax. 14.
 2 ÷ 1. 4 $\frac{C}{A} = \frac{C}{B}$. Ax. 14.
 ∴ e. 3. 4. 5 $A:C::B:C$. $C:A::C:B$. Def. 2. Q. E. D.

COROLLARY.

Hence Quantities directly proportional, are (Def. 3.) also so by Inversion.

PROP. VIII.

Of unequal Quantities A, B, the greater A, has a greater Ratio to a third C, than the lesser B; and this third C, has a greater Ratio to the lesser B, than the greater A.

That is, $\frac{A}{C} > \frac{B}{C}$, $\frac{C}{B} > \frac{C}{A}$.

DEMONSTRATION.

1 $A > B$. Hyp.
 ∴ e. 1. 2 $A \div C > B \div C$.
 Again 3 $B > A$. Hyp.
 ∴ e. 3. 4 $C \div B < C \div A$.
 ∴ e. 2. 4. 5 $\frac{A}{C} > \frac{B}{C}$, $\frac{C}{B} > \frac{C}{A}$. Q. E. D.

Qq

PROP.

PROP. IX.

Quantities A, B, which to one and the same Quantity C, have the same Proportion, are equal the one to the other; and if a Quantity C, have the same Proportion to other Quantities A, B, those Quantities are equal.

Hyp. 1 Suppose $A:C::B:C$.

I say, $A=B$.

DEMONSTRATION.

For 1 If A should be either $\square$ or $\neg B$.

Then 2 $\frac{A}{C}$ would be either $\square$ or $\neg \frac{B}{C}$. 8.

But 3 $\frac{A}{C} = \frac{B}{C}$. Hyp.

$\therefore$ 4 A is neither $\square$ nor $\neg B$.

$\therefore$ e. 4. 5 $A=B$. Ax. 4. Q. E. D.

Hyp. 2 Suppose $C:A::C:B$.

I say, $A=B$.

DEMONSTRATION.

For 1 If A should be either $\square$ or $\neg B$.

Then 2 $\frac{C}{A}$ would be either $\square$ or $\neg \frac{C}{B}$. 8.

But 3 $\frac{C}{A} = \frac{C}{B}$. Hyp.

$\therefore$ 4 A is neither $\square$ nor $\neg B$.

$\therefore$ e. 4. 5 $A=B$. Ax. 4. Q. E. D.

PROP.

PROP. X.

Of Quantities A, B, having Ratio to the same Quantity C, that Quantity A, which has the greater Ratio, is the greater Quantity; and that Quantity B, to which the same carries a greater Ratio, is the lesser Quantity.

Hyp. 1 Suppose $\frac{A}{C} \sqsupset \frac{B}{C}$.

I say, $A \sqsubset B$.

DEMONSTRATION.

For 1 If A should be = B.

Then 2 $\frac{A}{C}$ would be = $\frac{B}{C}$ 7.

But 3 $\frac{A}{C} \sqsubset \frac{B}{C}$. Hyp.

∴ 4 A is not = B.

Again 5 If A should be $\sqsupset$ B.

Then 6 $\frac{A}{C}$ would be $\sqsupset$ $\frac{B}{C}$ 8.

But 7 $\frac{A}{C} \sqsubset \frac{B}{C}$. Hyp.

∴ 8 A is not $\sqsupset$ B.

∴ 4. 8. 9 A is neither = nor $\sqsupset$ B.

∴ 9. 10 $A \sqsubset B$. Ax. 5. Q. E. D.

Q. 12

Suppose

Hyp. 2 Suppose $\frac{C}{B} \sqsubset \frac{C}{A}$.

I say, $B \sqsupset A$.

DEMONSTRATION.

For 1 If B should be $= A$.

Then 2 $\frac{C}{B}$ would be $= \frac{C}{A}$ 7.

But 3 $\frac{C}{B} \sqsubset \frac{C}{A}$ Hyp.

$\therefore$ 4 B is not $= A$.

Again 5 If B should be $\sqsubset A$.

Then 6 $\frac{C}{B}$ would be $\sqsupset \frac{C}{A}$ 8.

But 7 $\frac{C}{B} \sqsubset \frac{C}{A}$ Hyp.

$\therefore$ 8 B is not $\sqsubset A$.

$\therefore$ e. 4. 8. 9 B is neither $=$ nor $\sqsupset A$.

$\therefore$ e. 9. 10 $B \sqsupset A$. Ax. 6. Q. E. D.

PROP. XI.

Proportions which are one and the same to any Third, are also the same to one another.

Suppose $A:B::E:F$ and $C:D::E:F$.

I say, $A:B::C:D$.

DEMONSTRATION.

1 $\frac{A}{B} = \frac{E}{F}$ Hyp.

And 2 $\frac{C}{D} = \frac{E}{F}$ Hyp.

$\therefore$ e. 1. 2. 3 $\frac{A}{B} = \frac{C}{D}$ Ax. 1.

4 $A:B::C:D$. Def. 2. Q. E. D.

LEMMA.

LEMMA I.

If four Quantities $ARRR$, ARR , AR , A , or AR , A , BR , B , either continued or discontinued, are Proportional, that is, if $AR^3:AR^2::AR:A$. and $AR:A::BR:B$. the Product of the Extreams is equal to that of the Means.

That is, $AR^3 \times A = AR^2 \times AR$. and $AR \times B = BR \times A$.

DEMONSTRATION.

1 $AR^3:AR^2::AR:A$. and $AR:A::BR:B$. Def. 2.
Also 2 $AR^3 \times A = AR^2 \times AR$. and $AR \times B = BR \times A$.
Q. E. D.

LEMMA II.

Those four Quantities A , B , C , D , are Proportional, the Product of whose Extreams is equal to that of the Means.

PREPARATION.

Suppose $AD = BC$, and the Ratio of A to $B = X$, that is, $\frac{A}{B} = X$, then is $A = BX$; also suppose the Ratio of C to $D = Z$, that is, $\frac{C}{D} = Z$, then is $C = DZ$.

I say, $A:B::C:D$.

DEMONSTRATION.

1 $AD = BC$. Hyp.
And 2 $A = BX$.
∴ e. 1. 2. 3 $DBX = BC$. Sub.
Again 4 $C = DZ$.
∴ e. 3. 4. 5 $DBX = DBZ$. Sub.
5 ÷ DB . 6 $X = Z$. Ax. 14.
Lastly 7 $\frac{A}{B} = X$, $\frac{C}{D} = Z$. Hyp.
∴ e. 6. 7. 8 $\frac{A}{B} = \frac{C}{D}$. Sub.
∴ e. 8. 9 $A:B::C:D$. Def. 2. Q. E. D.

R r

PROP.

PROP. XII.

If several Quantities $A, AR, ARR, ARRR, ARRRR, ARRRRR$, &c. or A, AR, B, BR, C, CR , &c. either continued or discontinued, be Proportional, that is, if $A:AR::AR:AR^2$ and $A:AR::B:BR::C:CR$, as any one Antecedent is to its Consequent, so is the Sum of all the Antecedents, to the Sum of all the Consequents.

That is, $A:AR::A+AR+AR^2+AR^3+AR^4+AR^5$ and $A:AR::A+B+C:AR+BR+CR$.

DEMONSTRATION.

1 $A:AR::AR:AR^2::AR^2:AR^3$ and $A:AR::B:BR::C:CR$.
Def. 2.

And 2 $\frac{A+AR^2+AR^4}{A+B+C} \times AR = \frac{AR+AR^3+AR^5}{AR+BR+CR} \times A$ and
 $\frac{A+B+C}{A+B+C} \times AR = \frac{AR+BR+CR}{A+B+C} \times A$.

3 $A:AR::A+AR^2+AR^4:AR+AR^3+AR^5$ and $A:AR::A+B+C:AR+BR+CR$. Lem. 2. Q.E.D.

PROP. XIII.

If two Ratios be equal, and the one greater than a third Ratio, the other also will be greater than the same third Ratio.

Suppose $\frac{A}{B} = \frac{C}{D}$, also $\frac{C}{D} < \frac{E}{F}$.

I say, $\frac{A}{B} < \frac{E}{F}$.

DEMONSTRATION.

1 $\frac{A}{B} = \frac{C}{D}$. Hyp.

And 2 $\frac{C}{D} < \frac{E}{F}$. Hyp.

3 $\frac{A}{B} < \frac{E}{F}$: Ax. 3. Q.E.D.

PROP.

PROP. XIV.

In four proportional Quantities A, B, C, D, if the first A, be greater, equal, or less than the third C, then will the second B, be greater, equal, or less than the fourth D.

Hyp. 1 Suppose $A \sqsubset C$.

I say, $B \sqsubset D$.

DEMONSTRATION.

1 $A \sqsubset C$. Hyp.

∴ e. 1. 2 $\frac{A}{B} \sqsubset \frac{C}{B}$. 8.

Also 3 $\frac{A}{B} = \frac{C}{D}$. Hyp.

∴ e. 2. 3. 4 $\frac{C}{D} \sqsubset \frac{C}{B}$. Ax. 3.

∴ e. 4. 5 $B \sqsubset D$. 10. Q. E. D.

Hyp. 2 Suppose $A = C$.

I say, $B = D$.

DEMONSTRATION.

1 $A = C$. Hyp.

∴ e. 1. 2 $\frac{A}{B} = \frac{C}{B}$. 7.

Also 3 $\frac{A}{B} = \frac{C}{D}$. Hyp.

∴ e. 2. 3. 4 $\frac{C}{B} = \frac{C}{D}$. Ax. 1.

∴ e. 4. 5 $C:B::C:D$. Def. 2.

∴ e. 5. 6 $B = D$. 9. Q. E. D.

R r 2

Suppose

Hyp. 3 Suppose $A \sqsubset C$.

I say, $B \sqsubset D$.

DEMONSTRATION.

1 $A \sqsubset C$. Hyp.

∴ e. 1. 2 $\frac{A}{B} \sqsubset \frac{C}{B}$. 8.

Also 3 $\frac{A}{B} = \frac{C}{D}$. Hyp.

∴ e. 2. 3. 4 $\frac{C}{D} \sqsubset \frac{C}{B}$. Ax. 3.

∴ e. 4. 5 $B \sqsubset D$. 10. Q. E. D.

PROP. XV.

If of four Quantities AR , A , BR , B , the second A , be as often contained in the first AR , as is the fourth B , in the third BR , that is, if $\frac{AR}{A} = \frac{BR}{B}$, those four Quantities are Proportional.

That is, $AR : A :: BR : B$.

DEMONSTRATION.

1 $\frac{AR}{A} = \frac{BR}{B}$. Def. 2.

∴ e. 1. 2 $AR : A :: BR : B$. Def. 2. Q. E. D.

PROP.

PROP. XVI.

If four Quantities ARRR, ARR, AR, A, or AR, A, BR, B, either continued or discontinued, be Proportional, that is, if $AR:AR::AR:A$ and $AR:A::BR:B$, they are also so when Altern'd.

That is, $AR:BR::AR:A$ and $AR:BR::A:B$.

DEMONSTRATION.

- 1 $AR:AR::AR:A$ and $AR:A::BR:B$. Def. 2.
 2 $AR \times A = AR \times AR$ and $AR \times B = BR \times A$. Lem. 1.
 3 $AR:AR::AR:A$ and $AR:BR::A:B$. Lem. 2.
 Q. E. D.

PROP. XVII.

If four Quantities ARRR, ARR, AR, A, or AR, A, BR, B, either continued or discontinued, be Proportional, that is, if $AR:AR::AR:A$ and $AR:A::BR:B$, they are also so when Divided.

That is, $AR - AR:AR::AR - A:A$ and $AR - A:A::BR - B:B$.

DEMONSTRATION.

- 1 $AR:AR::AR:A$ and $AR:A::BR:B$. Def. 2.
 And 2 $AR - AR \times A = AR - A \times AR$ and $AR - A \times B = BR - B \times A$.
 3 $AR - AR:AR::AR - A:A$ and $AR - A:A::BR - B:B$. Lem. 2. Q. E. D.
 S f

PROP.

PROP. XVIII.

If four Quantities ARRR, ARR, AR, A, or AR, A, BR, B, continued or discontinued, are Proportional, that is, if $AR:AR::AR:A$. and $AR:A::BR:B$. they are also so when Compounded.

That is, $ARR + ARR::ARR::ARR + A:A$. and $ARR + A:A::BR + B:B$.

DEMONSTRATION.

- 1 $AR:AR::AR:A$. and $AR:A::BR:B$. Def. 2.
 And 2 $\frac{ARR + ARR}{ARR + A} \times A = \frac{ARR + A}{ARR + A} \times AR$. and $\frac{ARR + A}{ARR + A} \times B = \frac{BR + B}{BR + B} \times A$.
 ∴ c. 2. 3 $ARR + ARR::ARR::ARR + A:A$. and $ARR + A:A::BR + B:B$. Lem. 2. Q. E. D.

PROP. XIX.

If the whole $AB + BC$, be to the whole $DE + EF$, as the Part taken away AB , is to the Part taken away DE , then shall the Residue BC , be to the Residue EF , as the whole $AB + BC$, is to the whole $DE + EF$.

That is, $BC:EF::AB + BC:DE + EF$.

DEMONSTRATION.

- 1 $AB + BC:DE + EF::AB:DE$. Hyp.
 ∴ c. 1. 2 $AB + BC:AB::DE + EF:DE$. 16.
 ∴ c. 2. 3 $AB + BC - AB:AB::DE + EF - DE:DE$. 17.
 i.e. 4 $BC:AB::EF:DE$.
 ∴ c. 4. 5 $BC:EF::AB:DE$. 16.
 ∴ c. 5. 1. 6 $BC:EF::AB + BC:DE + EF$. 17. Q. E. D.
 COROL.

COROLLARY.

Because (Step. 2.) 'tis $AB + BC : AB :: DE + EF : DE$. and by inverting the Sixth Step, 'tis $AB + BC : BC :: DE + EF : EF$. therefore Quantities directly Proportional, are (Def. 7.) also so by Conversion.

PROP. XX.

If there be three Quantities A, B, C , and others D, E, F , equal to them in Number, which being taken two and two are in the same Ratio, and their Proportion Ordinate, that is, if $A : B :: D : E$ and $B : C :: E : F$. and if the first A be greater, equal, or less than the third C , then shall the fourth D be greater, equal, or less than the sixth F .

Hyp. 1 Suppose $A \square C$.

I say, $D \square F$.

DEMONSTRATION.

- 1 $B : C :: E : F$ Hyp.
 v.e. 1. 2 $C : B :: F : E$ Cor. 7.
 v.e. 2. 3 $\frac{C}{B} = \frac{F}{E}$ Def. 2.
 Again 4 $A \square C$ Hyp.
 v.e. 4. 5 $\frac{A}{B} \square \frac{C}{B}$ 8.
 v.e. 3. 5. 6 $\frac{A}{B} \square \frac{F}{E}$ Ax. 2.
 Also 7 $\frac{A}{B} = \frac{D}{E}$ Hyp.
 v.e. 6. 7. 8 $\frac{D}{E} \square \frac{F}{E}$ Ax. 3.
 v.e. 8. 9 $D \square F$ 10. Q. E. D.

S f 2

Suppose

Hyp. 2 Suppose $A = C$.

I say, $D = F$.

DEMONSTRATION.

1 $B:C::E:F$. Hyp.

And 2 $A = C$. Hyp.

∴ e. 1. 2. 3 $B:A::E:F$. Sub.

∴ e. 3. 4 $A:B::F:E$. Cor. 7.

Allo 5 $A:B::D:E$. Hyp.

∴ e. 5. 4. 6 $D:B::F:E$. 11.

∴ e. 6. 7 $D = F$. 9. Q. E. D.

Hyp. 3 Suppose $A \sqsubset C$.

I say, $D \sqsubset F$.

DEMONSTRATION.

1 $B:C::E:F$. Hyp.

∴ e. 1. 2 $C:B::F:E$. Cor. 7.

∴ e. 2. 3 $\frac{C}{B} = \frac{F}{E}$. Def. 2.

Again 4 $A \sqsubset C$. Hyp.

∴ e. 4. 5 $\frac{A}{B} \sqsubset \frac{C}{B}$. 8.

∴ e. 3. 5. 6 $\frac{A}{B} \sqsubset \frac{F}{E}$. Ax. 2.

Allo 7 $\frac{A}{B} = \frac{D}{E}$. Hyp.

∴ e. 6. 7. 8 $\frac{D}{E} \sqsubset \frac{F}{E}$. Ax. 3.

∴ e. 8. 9 $D \sqsubset F$. 10. Q. E. D.

PROP.

PROP. XXI.

If there be three Quantities A, B, C, and others D, E, F, equal to them in Number, which taken two and two are in the same Ratio, and their Proportion Inordinate; that is, if $A:B::E:F.$ and $B:C::D:E.$ and if the first A be greater, equal, or less than the third C, then is the fourth D greater, equal, or less than the sixth F.

Hyp. 1 Suppose $A \sqsubset C.$

I say, $D \sqsubset F.$

DEMONSTRATION.

1 $B:C::D:E.$ Hyp.

∴ c. 1. 2 $C:B::E:D.$ Cor. 7.

∴ e. 2. 3 $\frac{C}{B} = \frac{E}{D}.$ Def. 2.

Again 4 $A \sqsubset C.$ Hyp.

∴ e. 4. 5 $\frac{A}{B} \sqsubset \frac{C}{B}.$ 8.

∴ c. 3. 5. 6 $\frac{A}{B} \sqsubset \frac{E}{D}.$ Ax. 2.

Also 7 $\frac{A}{B} = \frac{E}{F}.$ Hyp.

∴ c. 8. 7. 8 $\frac{E}{F} \sqsubset \frac{E}{D}.$ Ax. 3.

∴ c. 8. 9 $D \sqsubset F.$ 10. Q. E. D.

Hyp. 2 Suppose $A = C.$

I say, $D = F.$

T t

DE

DEMONSTRATION.

- $\therefore$ e. 1. 1 B : C :: D : E. Hyp.
 Again 2 C : B :: E : D. Cor. 7.
 $\therefore$ e. 2. 3 A = C. Hyp.
 Also 4 A : B :: E : D. Sub.
 $\therefore$ e. 4. 5 A : B :: E : F. Hyp.
 6 E : D :: E : F. 11.
 $\therefore$ e. 6. 7 $\frac{E}{D} = \frac{E}{F}$. Def. 2.
 $\therefore$ e. 7. 8 $\therefore = \therefore$. 9. Q. E. D.

Hyp. 3 Suppose A $\sqsubset$ C.

I say, $\therefore \sqsubset \therefore$.

DEMONSTRATION.

- $\therefore$ e. 1. 1 B : C :: D : E. Hyp.
 $\therefore$ e. 1. 2 C : B :: E : D. Cor. 7.
 $\therefore$ e. 2. 3 $\frac{C}{B} = \frac{E}{D}$. Def. 2.
 Again 4 A $\sqsubset$ C. Hyp.
 $\therefore$ e. 4. 5 $\frac{A}{B} \sqsubset \frac{C}{B}$. 8.
 $\therefore$ e. 3. 5. 6 $\frac{A}{B} \sqsubset \frac{E}{D}$. Ax. 2.
 Also 7 $\frac{A}{B} = \frac{E}{F}$. Hyp.
 $\therefore$ e. 6. 7. 8 $\frac{E}{F} \sqsubset \frac{E}{D}$. Ax. 3.
 $\therefore$ e. 8. 9 $\therefore \sqsubset \therefore$. 10. Q. E. D.

PROP.

PROP. XXII.

If there be a certain Number of Quantities A, B, C, in one Rank, and others D, E, F, equal to them in Number in another, which taken two and two are in the same Ratio, and their Proportion Ordinate, that is, if $A:B::D:E$. and $B:C::E:F$. the Ratio of the two Extrems of the one Rank, shall be equal to the Ratio of the two Extrems of the other.

That is, $A:C::D:F$.

DEMONSTRATION.

- | | | | |
|------------|---|------------|--------------|
| | 1 | $B:C::E:F$ | Hyp. |
| ∴ e. 1. | 2 | $B:E::C:F$ | 16. |
| Also | 3 | $A:B::D:E$ | Hyp. |
| ∴ e. 3. | 4 | $A:D::B:E$ | 16. |
| ∴ e. 4. 2. | 5 | $A:D::C:F$ | 11. |
| ∴ e. 5. | 6 | $A:C::D:F$ | 16. Q. E. D. |

PROP. XXIII.

If there be a certain Number of Quantities A, B, C, in one Rank, and others D, E, F, equal to them in Number in another, which taken two and two are in the same Ratio, and their Proportion Inordinate; that is, if $A:B::E:F$. and $B:C::D:E$. the Ratio of the two Extrems of the one Rank, is equal to the Ratio of the two Extrems of the other.

That is, $A:C::D:F$.

DEMONSTRATION.

- 1 $A : B :: E : F$. Hyp.
 2 $AF = BE$. Lem. 1.
 3 $B : C :: D : E$. Hyp.
 4 $CD = BE$. Lem. 1.
 5 $AF = CD$. Ax. 1.
 6 $A : C :: D : F$. Lem. 2. Q. E. D.

PROP. XXIV.

If of Six Quantities A, B, C, D, E, F , the first A , have the same Ratio to the second B , which the third C , has to the fourth D , that is, if $\frac{A}{B} = \frac{C}{D}$; and the fifth E , have the same Ratio to the second B , which the sixth F , has to the fourth D , that is, if $\frac{E}{B} = \frac{F}{D}$; then shall the first compounded with the fifth, have the same Ratio to the second, which the third compounded with the sixth has to the fourth.

That is, $A + E : B :: C + F : D$.

DEMONSTRATION.

- 1 $\frac{A}{B} = \frac{C}{D}$. Hyp.
 And 2 $\frac{E}{B} = \frac{F}{D}$. Hyp.
 3 $\frac{A + E}{B} = \frac{C + F}{D}$. Ax. II.
 4 $A + E : B :: C + F : D$. Def. 2. Q. E. D.

PROP.

PROP. XXV:

If four Quantities A, B, C, D, are proportional, that is, if $A:B::C:D$, the Sum $A+D$, of the greatest A, and the least D, shall be greater than the Sum $B+C$, of the other two.

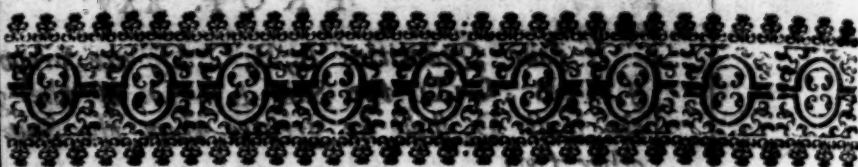
That is, $A+D > B+C$.

DEMONSTRATION.

- | | | | |
|------------|---|-------------------------------|--|
| | 1 | $A:B::C:D$. Hyp. | |
| ∴ e. 1. | 2 | $A-B:B::C-D:D$. 17. | |
| ∴ e. 2. | 3 | $B:A-B::D:C-D$. Cor. 7. | |
| Now | 4 | $B < D$. Hyp. | |
| ∴ e. 3. 4. | 5 | $A-B < C-D$. 14. | |
| $5+B+D$ | 6 | $A-B+B+D < C-D+B+D$. Ax. 15. | |
| i. e. | 7 | $A+D < B+C$. Q. E. D. | |

The End of the Fifth Book.





EUCLID'S ELEMENTS,

Demonstrated after a New, Plain, and Easy
METHOD.

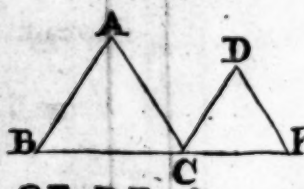
BOOK VI.

DEFINITIONS.



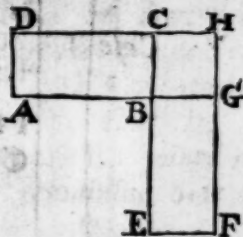
I Similar Right-lined Figures, are such whose several Angles are equal one to the other, and which have also their Sides about the equal Angles, proportional.

Thus the two Right-lined Figures ABC , DCE , are similar, because $\angle B = \angle DCE$, and $AB:BC::DC:CE$; the $\angle A = \angle D$, and $AB:AC::DC:DE$; the $\angle ACB = \angle E$, and $BC:AC::CE:DE$.

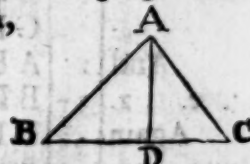


- 2 **Reciprocal Figures** are such, as have Sides that may be so compared, as that the Antecedent of one Ratio, and the Consequent of the other, is to be found in the same Figure.

Thus the two Figures BD, BF, are **Reciprocal**, because as the Side AB, is to the Side BG, so is the Side EB, to the Side BC.



- 3 The **Altitude** of any Figure ABC, is a perpendicular Line AD, drawn from the Top A, to the Base BC.



- 4 A **Ratio** is said to be compounded of two Ratio's, when the Quantities of the Ratio's being multiplied the one into the other, do produce any Ratio.

Thus the Ratio of A to C, that is $\frac{A}{C}$ is compounded of the Ratio's of A to B, and B to C, that is, of $\frac{A}{B}$ and $\frac{B}{C}$.

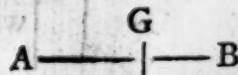
$$\text{Because } \frac{A}{B} \times \frac{B}{C} = \frac{AB}{BC} = \frac{A}{C}.$$

- 5 In any given Quantities A, B, C, D, the Ratio of the Extreams A, D, that is $\frac{A}{D}$, is compounded of all the in-

termediate ones $\frac{A}{B}, \frac{B}{C}, \frac{C}{D}$. For $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} =$

$$\frac{ABC}{BCD} = \frac{A}{D}.$$

- 6 A **Right-line AB**, is said to be cut according to **extream and mean Ratio**, when as the whole AB, is to the greater Segment AG, so is the greater Segment AG, to the less GB.



U u 2

PROP.

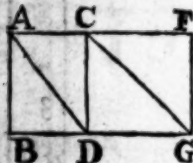
PROPOSITION I.

Parallelograms and Triangles, which have the same Height, are in Proportion one to the other, as their Bases.

Case 1 If the $\square$ ABDC, CDGF, and $\triangle$ ABD, GDC, be Right-angled.

I say, $BD:DC::\square ABDC:\square CDGF::\triangle ABD:\triangle CDG$.

DEMONSTRATION.



$BD \times AB = \square ABDC$; $DG \times CD = \square CDGF$. *Lem. 1. 1.*

And $AB = CD$. *34. 1.*

$\therefore$ *e. 1. 2.* $BD \times AB = \square ABDC$; $DG \times AB = \square CDGF$. *Sub.*

Again $BD:DG::BD \times AB:DG \times AB$. *Lem. 2. 5.*

$\therefore$ *e. 4. 3.* $BD:DG::\square ABDC:\square CDGF$. *Sub.*

Lastly $\square ABDC = 2\triangle ABD$; $\square CDGF = 2\triangle CDG$. *41. 1.*

$\therefore$ *e. 5. 6.* $BD:DG::2\triangle ABD:2\triangle CDG$. *Sub.*

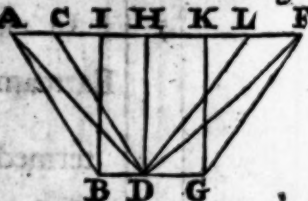
$\therefore$ *e. 7.* $BD:DG::\triangle ABD:\triangle CDG$.

$\therefore$ *e. 5. 8.* $BD:DC::\square ABDC:\square CDGF::\triangle ABD:\triangle CDG$. *11. 5. Q. E. D.*

Case. 2 If the $\square$ ABDC, LDGF, and $\triangle$ ABD, FDG, be Oblique-angled.

Erect Right-angled $\square$ BDHI, DGKH, upon the same Bases, between the same Parallels, or in the same Height with them; so (35. 1.) will those Right-angled $\square$ be equal to the Oblique-Angled ones.

Since therefore the Right-angled $\square$ BDHI, DGKH, are (*Case 1.*) in Proportion one to another, as their Bases; the Oblique-angled $\square$ ABDC, LDGF, their Equals; as also the Oblique-angled $\triangle$ ABD, FDG, (which (41. 1.) are the Halves of the Right-angled $\square$ BDHI, DGKH) will be in Proportion one to another, as their Bases. Q. E. D.



P R O P.

PROP. II.

If to one Side BC , of a Triangle ABC , be drawn a parallel Right-line DE , the same shall cut the Sides of the Triangle proportionally, so that $AD:DB::AE:EC$. And if the Sides of the Triangle be cut proportionally, that is, if $AD:DB::AE:EC$. then a Right-line DE , joined at the Sections D, E , shall be parallel to the remaining Side of the Triangle BC .

Hyp. 1 Suppose $DE \parallel BC$,

I say, $AD:DB::AE:EC$.



DEMONSTRATION.

$AD:DB::\triangle ADE:\triangle BDE$. 1.

And $\triangle BDE = \triangle CDE$. 37. 1.

"c. 1. 2. 3 $AD:DB::\triangle ADE:\triangle CDE$. Sub.

Also 4 $AE:EC::\triangle ADE:\triangle CDE$. 1.

"c. 3. 4. 5 $AD:DB::AE:EC$. 11. 5. Q. E. D.

Hyp. 2 Suppose $AD:DB::AE:EC$.

I say, $DE \parallel BC$.

DEMONSTRATION.

1 $AD:DB::AE:EC$. Hyp.

Also 2 $AD:DB::\triangle ADE:\triangle BDE$. 1.

"c. 1. 2. 3 $AE:EC::\triangle ADE:\triangle BDE$. 11. 5.

Also 4 $AE:EC::\triangle ADE:\triangle CDE$. 1.

"c. 3. 4. 5 $\triangle ADE:\triangle BDE::\triangle ADE:\triangle CDE$. 11. 5.

"c. 5. 6 $\triangle BDE = \triangle CDE$. 9. 5.

"c. 6. 7 $DE \parallel BC$. 37. 1. Q. E. D.

X x

SCHOLIUM

S C H O L I U M.

If there be drawn many Parallels to One Side of any Triangle, then all the Segments of the Sides shall be Proportional ; as is easily deducible from the precedent.

P R O P. III.

If an Angle BAC , of a Triangle BAC , be bisected, and the Right-line AD , that bisects the Angle, cut the Base also ; then shall the Segments of the Base have the same Ratio that the other Sides of the Triangle have, so that $BD : DC :: BA : AC$. And if the Segments of the Base have the same Ratio, that the other Sides of the Triangle have, that is, if $BD : DC :: BA : AC$. then a Right-line AD , drawn from the Top A , to the Section D , shall bisect that Angle BAC of the Triangle.

Hyp. 1

Suppose $\angle BAD = \angle CAD$.

P R E P A R A T I O N.

Produce BA ; and making $AE = AC$, join EC .

I say, $BD : DC :: BA : AC$.

D E M O N S T R A T I O N.

- | | |
|---|--|
| 1 | $AE = AC$. <i>Const.</i> |
| 2 | $\angle E = \angle ACE$. 5. 1. |
| 3 | $\angle BAC = \angle E + \angle ACE$. 32. 1. |
| 4 | $\angle BAC = \angle ACE + \angle ACE$. <i>Sub.</i> |
| 5 | $\angle BAC = 2 \angle ACE$. |
| 6 | $\frac{1}{2} \angle BAC = \angle ACE$. <i>Am. 14.</i> |
| 7 | $\angle BAC = \angle CAD$. <i>Hyp.</i> |

∴ e. 1.

Again

7. 1. e. 2. 3.

i. e.

5 ÷ 2

Also



∠A

- $\therefore$ e. 6. 7. 8 $\angle ACE = \angle CAD$. Ax. 8.
 $\therefore$ e. 8. 9 $AD \parallel EC$. 27. 1.
 $\therefore$ e. 9. 10 $BD : DC :: BA : AE$. 2.
 $\therefore$ e. 10. 1. 11 $BD : DC :: BA : AC$. Sub. Q. E. D.

Hyp. 2 Suppose $BD : DC :: BA : AC$.

I say, $\angle BAD = \angle CED$.

DEMONSTRATION.

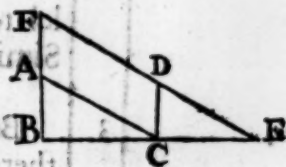
- $\therefore$ e. 1. 2 $AC = AE$. Const.
 Again 3 $BD : DC :: BA : AC$. Hyp.
 $\therefore$ e. 1. 3. 4 $BD : DC :: BA : AE$. Sub.
 $\therefore$ e. 4. 5 $AD \parallel EC$. 2.
 $\therefore$ e. 5. 6 $\angle BAD = \angle E$. 29. 1.
 $\therefore$ e. 2. 6. 7 $\angle ACE = \angle BAD$. Ax. 1.
 Also e. 5. 8 $\angle ACE = \angle CAD$. 29. 1.
 $\therefore$ e. 7. 8. 9 $\angle BAD = \angle CED$. Ax. 1. Q. E. D.

PROP. IV.

Of Equiangular Triangles ABC , DCE , the Sides which are about the equal Angles ACB and DEC , B and D , CE , BAC and CDE , are proportional.

PREPARATION.

Set the Side BC , in a direct Line to the Side CE , and produce BA and DE till they meet in F .



I say, $BC : CE :: CE : DE$. $BC : BA :: CE : CD$.

X x 2

DE-

DEMONSTRATION.

- 1 $\angle B = \angle DCE, \angle ACB =$
 $\angle DEG.$ *Hyp.*
- 2 $BF \parallel CD, AC \parallel FE.$ 28. 1.
- 3 $ACDF$ is a $\square.$ *Def. 35. 1.*
- 4 $AC = FD, AF = CD.$ 34. 1.
- 5 $BC : CE :: FD : DE; BC : CE :: BA : AF.$ 2.
- 6 $BC : CE :: AC : DE; BC : CE :: BA : CD.$ *Sub.*
- 7 $BC : AC :: CE : DE; BC : BA :: CE : CD.$ 16. 5.
- 8 $AC : DE :: BA : CD.$ 11. 5.
- 9 $BA : AC :: CD : DE.$ *Cor. 7. 5.*
- 10 $BC : AC :: CD : DE. Q. E. D.$



S C H O L I A.

1 Because Equiangular Triangles have also their Sides about the equal Angles proportional ; therefore Equiangular Triangles are likewise (*Def. 1.*) Similar.

2 Because (*Hyp.*) $\angle ACB = \angle DEC$, $\angle B$ is common, and consequently (*Cor. 2. 32. 1.*) $\angle BAC = \angle F$; therefore if in a Triangle FBE , there be drawn AC , a parallel to one Side FE , the Triangle ABC , shall (*Scho. 1.*) be Similar to the whole FBE .

3 Because (*Step. 6.*) $BC : CE :: AC : DE :: BA : CD$: therefore the Sides subtended under the equal Angles are Homologous, or of like Ratio.

PROP.

PROP. V.

If two Triangles ABC, DEF , have their Sides proportional, that is, if $AB:BC::DE:EF$, and $AC:BC::DF:EF$, and $AB:AC::DE:DF$, those Triangles are Equiangular, and have those Angles equal, under which are subtended the Homologous Sides.

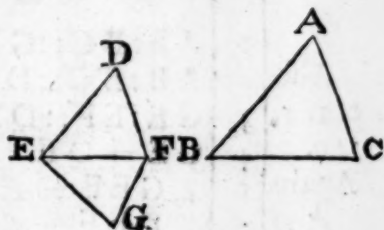
PREPARATION.

At the Side EF , make (23. 1.) $\angle GEF = \angle B$, $\angle GFE = \angle C$, whence (Cor. 2. 32. 1.) $\angle G = \angle A$, and consequently the Triangles ABC, GEF , are Equiangular.

I say, $\angle A = \angle D$. $\angle B = \angle DEF$. $\angle C = \angle DFE$.

DEMONSTRATION.

- | | | | |
|--------------------------|----|---|--|
| | 1 | $AB:BC::GE:EF$. 4. | |
| Also | 2 | $AB:BC::DE:EF$. Hyp. | |
| $\therefore$ e. 1. 2. | 3 | $GE:EF::DE:EF$. 11. 5. | |
| $\therefore$ e. 3. | 4 | $GE = DE$. 9. 5. | |
| Again | 5 | $AC:BC::GF:EF$. 4. | |
| Also | 6 | $AC:BC::DF:EF$. Hyp. | |
| $\therefore$ e. 5. 6. | 7 | $GF:EF::DF:EF$. 11. 5. | |
| $\therefore$ e. 7. | 8 | $GF = DF$. 9. 5. | |
| Lastly | 9 | EF is common. | |
| $\therefore$ e. 4. 8. 9. | 10 | $GE = DE, GF = DF, EF$ is common. | |
| $\therefore$ e. 10. | 11 | $\angle G = \angle D, \angle GEF = \angle DEF, \angle GFE = \angle DFE$. 8. 1. | |
| Also | 12 | $\angle G = \angle A, \angle GEF = \angle B, \angle GFE = \angle C$. Const. | |
| $\therefore$ e. 12. 11. | 13 | $\angle A = \angle D, \angle B = \angle DEF, \angle C = \angle DFE$. Ax. 1. | |
- Q. E. D.



PROP. VI.

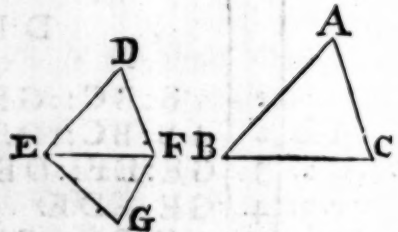
If two Triangles ABC, DEF, have one Angle B, equal to one Angle DEF, and the Sides about the equal Angles B, DEF, proportional, that is, if $AB:BC::DE:EF$; then those Triangles ABC, DEF, are Equiangular, and have those Angles equal, under which are subtended the Homologous Sides.

PREPARATION.

At the Side EF, make (23. 1.) $\angle GEF = \angle B$, $\angle GFE = \angle C$, whence (Cor. 2. 32. 1.) $\angle G = \angle A$, and consequently the Triangles ABC, GEF, are Equiangular.

I say, $\angle A = \angle D$. $\angle B = \angle DEF$. $\angle C = \angle DFE$.

DEMONSTRATION.



- | | | |
|--------------------------|----|---|
| | 1 | $AB:BC::GE:EF$. 4. |
| Also | 2 | $AB:BC::DE:EF$. Hyp. |
| $\therefore$ e. 1. 2. | 3 | $GE:EF::DE:EF$. 11. 5. |
| $\therefore$ e. 3. | 4 | $GE = DE$. 9. 5. |
| Again | 5 | $\angle GEF = \angle B$. Const. |
| Also | 6 | $\angle DEF = \angle B$. Hyp. |
| $\therefore$ e. 5. 6. | 7 | $\angle GEF = \angle DEF$. Ax. 1. |
| Lastly | 8 | EF is common. |
| $\therefore$ e. 4. 8. 7. | 9 | $GE = DE$, EF is common, $\angle GEF = \angle DEF$. |
| $\therefore$ e. 9. | 10 | $\angle G = \angle D$, $\angle GEF = \angle DEF$, $\angle GFE = \angle DFE$. 4. 1. |
| Also | 11 | $\angle G = \angle A$, $\angle GEF = \angle B$, $\angle GFE = \angle C$. Const. |
| $\therefore$ e. 11. 10. | 12 | $\angle A = \angle D$. $\angle B = \angle DEF$. $\angle C = \angle DFE$. Ax. 1. |
- Q. E. D.

The Seventh Proposition is needless.

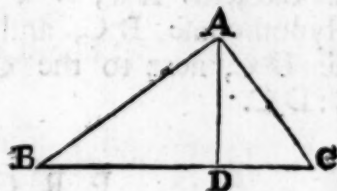
PROP.

PROP. VIII.

A Perpendicular AD, let fall from the Right-angle BAC, of a Right-angled Triangle ABC, upon the opposite Side BC, divides the Triangle ABC, into two others ADB, ADC, similar to its self.

That is, $\triangle ABC$, $\triangle ADB$, $\triangle ADC$, are Similar.

DEMONSTRATION.



- | | | |
|----------------------------|----|--|
| | 1 | ADB is a $r\angle$. Def. 10. 1. |
| $\therefore e. 1.$ | 2 | $\angle BAD + \angle B = r\angle$. Cor. 3. 32. 1. |
| Also | 3 | BAC is a $r\angle$. Hyp. |
| $\therefore e. 3.$ | 4 | $\angle C + \angle B = r\angle$. Cor. 3. 32. 1. |
| $\therefore e. 2. 4.$ | 5 | $\angle BAD + \angle B = \angle C + \angle B$. Ax. 1. |
| 5 — $\angle B$. | 6 | $\angle BAD = \angle C$. Ax. 12. |
| Again | 7 | ADC is a $r\angle$: Def. 10. 1. |
| $\therefore e. 7.$ | 8 | $\angle C + \angle CAD = r\angle$. Cor. 3. 32. 1. |
| $\therefore e. 4. 8.$ | 9 | $\angle C + \angle B = \angle C + \angle CAD$. Ax. 1. |
| 9 — $\angle C$. | 10 | $\angle B = \angle CAD$. Ax. 12. |
| Lastly $e. 3. 1. 7.$ | 11 | $\angle BAC = \angle ADB = \angle ADC$: Ax. 2. 1. |
| $\therefore e. 11. 10. 6.$ | 12 | $\angle BAC = \angle ADB = \angle ADC$; $\angle B = \angle CAD$; $\angle B$
$AD = \angle C$. |
| $\therefore e. 12.$ | 13 | $\triangle ABC$, $\triangle ADB$, $\triangle ADC$, are Similar. Scho. 1. 4.
Q. E. D. |

Q. E. D.

COROL.

COROLLARIES.

1 In any Right angled Triangle, a Perpendicular being let fall from the Right-angle upon the Hypothenuſe, will (Def. 1.) be a mean Proportional between the Segments of the Hypothenuſe. That is, $BD:AD::AD:DC$.

2 The Baſe BA , is a mean Proportional, between the Hypothenuſe BC , and that Segment of the Hypothenuſe BD , next to the Baſe. That is, $BC:BA::BA:BD$.

3 The Cathetus AC , is a mean Proportional, between the Hypothenuſe BC , and that Segment of the Hypothenuſe DC , next to the Cathetus. That is, $BC:AC::AC:DC$.

PROP. IX.

From a Right-line given AB , to cut off any Part required, as one third AG .

PREPARATION.

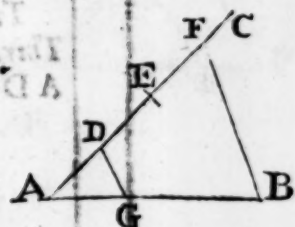
From the Point A , draw an infinite Line AC any-
 wiſe, in which take (3. 1.) any three equal Parts AD ,
 DE , EF ; then join FB , to which from D , draw (31.
 1.) the Parallel DG .

I ſay, $AG = \frac{1}{3} AB$.

DEMON-

DEMONSTRATION.

1. $FD:AD::BG:AG.$ 2.
2. $FD+AD:AD::BG+AG:AG.$ 18. 5.
- i. e. 3. $AF:AD::AB:AG.$
- e. 3. 4. $AD:AF::AG:AB.$ Cor. 7. 5.
- But 5. $AD=AF.$ Const.
- e. 4. 5. 6. $AG=AB.$ Q. E. F. & D.



Join B D, and from A B produced, take B C = A D. Then C, F, B, D, will be produced meet in E.

To divide a given Line A D, in the same manner as another given Line A E is divided, so that $AC:CE::AB:BD.$

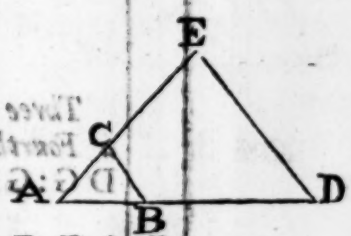
DEMONSTRATION.
PREPARATION.

Join the two given Lines A D, A E, at any Angle D A E; then having joined E D, from the Point of Division C, draw (1. 11) $CB \parallel ED.$

I say, $AC:CE::AB:BD.$

DEMONSTRATION.

1. $CB \parallel ED.$ Const.
- e. 1. 2. $AC:CE::AB:BD.$ 2. Q. E. F. & D.



PROP.

PROP. XI.

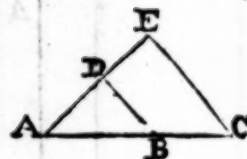
Two Right-lines AB, AD, being given, to find out a Third DE, in Proportion to them, so that $AB:AD::AD:DE$.

PREPARATION.

Join BD, and from AB produced, take $BC=AD$. Thro' C, draw (31. 1.) $CE \parallel BD$; with which let AD produced meet in E.

I say, $AB:AD::AD:DE$.

DEMONSTRATION.



1. $AB:BC::AD:DE$. 2.
 And 2. $BC=AD$. Const.
 3. $AB:AD::AD:DE$. Sub. Q. E. F. & D.

PROP. XII.

Three Right-lines DE, EF, DG, being given, to find out a Fourth GH, in Proportion to them, so that $DE:EF::DG:GH$.

PREPARATION.

Join EG, and thro' F, draw (31. 1.) $FH \parallel EG$; with which let DG produced meet in H.

I say,

I say, $DE:EF::DC:CH$.

DEMONSTRATION.



$FH \parallel EG$. *Const.*

$DE:EF::DC:CH$. 1. Q. E. F. & D.

PROP. XIII.

Two Right-lines BD, DC, being given, to find out a mean Proportional AD, so that $BD:AD::AD:DC$.

PREPARATION.

Upon the whole Line BC, as a Diameter, describe a Semicircle BAC, and having from D, erected a Perpendicular AD, meeting with the Periphery in A, draw AB, AC, so (31. 3.) shall BAC be a Right-angle.

I say, $BD:AD::AD:DC$.

DEMONSTRATION.



BAC is a $\angle$. *Const.*

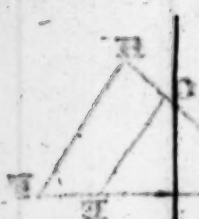
$BD:AD::AD:DC$. Cor. 1. 8. Q. E. F. & D.

COROLLARY.

Hence, a Right-line drawn in a Circle, from any Point of the Diameter perpendicular, and extended to the Circumference, is a mean Proportional betwixt the two Segments of the Diameter.

Z z

PROP.



PROP. XIV.

Equal Parallelograms BD, BF, having one Angle ABC, equal to one EBG, have the Sides about the equal Angles reciprocal, so that $AB:BG::EB:BC$. And those Parallelograms BD, BF, which have one Angle ABC, equal to one EBG, and the Sides which are about the equal Angles reciprocal, so that $AB:BG::EB:BC$, are equal.

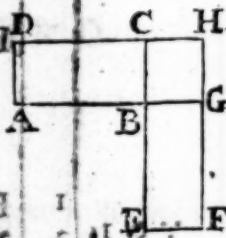
Hyp. 1. Suppose $\square BD = \square BF$, $\angle ABC = \angle EBG$.

PREPARATION

Let the Sides AB, BG, about the equal Angles, be set in a straight Line, so (Scho. 15. 1.) shall EBC be likewise a straight Line; also let DC, FG, be produced till they meet in H.

I say, $AB:BG::EB:BC$.

DEMONSTRATION



1. $AB:BG::\square BD:\square BH$.
- And 2. $\square BD = \square BF$.
- ∴ e. 1. 2. 3. $AB:BG::\square BF:\square BH$.
- Also 4. $EB:BC::\square BF:\square BH$.
- ∴ e. 3. 4. 5. $AB:BG::EB:BC$.

Hyp. 2. Suppose $\angle ABC = \angle EBG$, $AB:BG::EB:BC$.

I say, $\square BD = \square BF$.

DEMON.

DEMONSTRATION.

- 1 $AB:BG::EB:BC$ Hyp.
 Also 2 $AB:BG::\square BD:\square BH$ I.
 ∴ e. 1. 2. 3 $EB:BC::\square BD:\square BH$ II. 5.
 Also 4 $EB:BC::\square BF:\square BH$ I.
 ∴ e. 3. 4. 5 $\square BD:\square BH::\square BF:\square BH$ II. 5.
 ∴ e. 5. 6 $\square BD = \square BF$ 9. 5. Q. E. D.

PROP. XV.

Equal Triangles ACB, DEB , having one Angle ABC equal to one DBE , have their Sides which are about the equal Angles reciprocal, so that $AB:BE::DB:BC$. And those Triangles that have one Angle ABC , equal to one DBE , and the Sides which are about the equal Angles reciprocal, so that $AB:BE::DB:BC$, are equal.

Hyp. 1 Suppose $\triangle ACB = \triangle DEB$, $\angle ABC = \angle DBE$.

PREPARATION.

Let the Sides AB, BE , which are about the equal Angles, be set in a straight Line, so (Scho. 15. 1.) shall DBC be likewise a straight Line; also join CE .

I say, $AB:BE::DB:BC$.

DEMONSTRATION.

- 1 $\triangle ACB = \triangle DEB$ Hyp.
 ∴ e. 1. 2 $\triangle ACB:\triangle ECB::\triangle DEB:\triangle ECB$ 7. 5.
 Also 3 $\triangle ACB:\triangle ECB::AB:BE$ I.
 ∴ e. 3. 2. 4 $AB:BE::\triangle DEB:\triangle ECB$ II. 5.
 Also 5 $DB:BC::\triangle DEB:\triangle ECB$ I.
 ∴ e. 4. 5. 6 $AB:BE::DB:BC$ II. 5. Q. E. D.



A a a

Suppose

Hyp. 2 Suppose $\angle ABC = \angle DBE$, $AB:BE::DB:BC$.

I say, $\triangle ACB = \triangle ECB$.

DEMONSTRATION.

1 $AB:BE::DB:BC$. Hyp.
 Also 2 $AB:BE::\triangle ACB:\triangle ECB$. 1.
 ∴ e. 1. 2. 3 $DB:BC::\triangle ACB:\triangle ECB$. 11. 5.
 Also 4 $DB:BC::\triangle DEB:\triangle ECB$. 1.
 ∴ e. 3. 4. 5 $\triangle ACB:\triangle ECB::\triangle DEB:\triangle ECB$. 11. 5.
 ∴ e. 5. 6 $\triangle ACB = \triangle ECB$. 9. 5. Q. E. D.



PROP. XVI

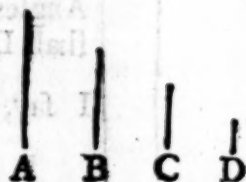
If four Lines A, B, C, D, are Proportional, that is, if $A:B::C:D$, the Rectangle of the two Extremes A, D, is equal to the Rectangle of the two Means B, C. And if the Rectangle of the two Extremes A, D, be equal to that of the two Means B, C, the four Lines are Proportional.

Hyp. 1 Suppose $A:B::C:D$.

I say, $AD = BC$.

DEMONSTRATION.

1 $A:B::C:D$. Hyp.
 ∴ e. 1. 2 $AD = BC$. Lem. 1. 5. Q. E. D.



Hyp. 2 Suppose $AD = BC$.

I say, $A:B::C:D$.

DEMONSTRATION.

1 $AD = BC$. Hyp.
 ∴ e. 1. 2 $A:B::C:D$. Lem. 2. 5. Q. E. D.

PROP.

PROP. XVII.

If three Lines A, B, C, be continually Proportional, that is, if $A:B::B:C$, the Rectangle of the two Extreams A, C, is equal to the Square of the Mean B. And if the Rectangle of the two Extreams A, C, be equal to the Square of the Mean B, the three Lines are Proportional.

Hyp. 1 Suppose $A:B::B:C$.

PREPARATION.

Make $D = B$.

I say, $AC = BD$.

DEMONSTRATION.

1 $A:B::B:C$ Hyp.
And 2 $D=B$ Const.
∴ e. 1. 2. 3 $A:B::D:C$ Sub.
∴ e. 3. 4 $AC = BD$ 16.
∴ e. 2. 4. 5 $AC = BD$ Sub.
i. e. 6 $AC = BD$ Q. E. D.



Hyp. 2 Suppose $AC = BD$.

I say, $A:B::B:C$.

DEMONSTRATION.

1 $AC = BD$ Hyp.
i. e. 2 $AC = BD$
Also 3 $D=B$ Const.
∴ e. 2. 3. 4 $AC = BD$ Sub.
∴ e. 4. 5 $A:B::D:C$ 16.
∴ e. 3. 5. 6 $A:B::B:C$ Sub. Q. E. D.

A 2 2 2

PROP.

PROP. XVIII.

Upon a Right-line given AB, to describe a Right-lined Figure ABGH, similar to a Right-lined Figure given CFDE.

PREPARATION.

Make (23. 1.) $\angle ABH = \angle CFE$, $\angle GBH = \angle DFE$, $\angle BHA = \angle FEC$, $\angle BHG = \angle FED$, so (Cor. 2. 32. 1.) is $\angle A = \angle C$, $\angle G = \angle D$, and consequently the $\triangle ABH$, $\triangle CFE$, as also the $\triangle GBH$, $\triangle DFE$, are Equiangular; also (Ax. 11.) $\angle ABH + \angle GBH = \angle CFE + \angle DFE$; $\angle BHA + \angle BHG = \angle FEC + \angle FED$; that is, $\angle ABG = \angle CFD$, $\angle AHG = \angle CED$.

I say, ABGH, CFDE, are Similar.

DEMONSTRATION.



- | | | | |
|------------------|---|---|--------|
| | 1 | AH: BH :: CE: FE; BH: GH :: FE: DE. | 4. |
| ∴ e. 1. | 2 | AH: GH :: CE: DE. | 22. 5. |
| Also | 3 | AB: BH :: CF: FE; BH: BG :: FE: FD. | 4. |
| ∴ e. 3. | 4 | AB: BG :: CF: FD. | 22. 5. |
| Again | 5 | AB: AH :: CF: CE; BG: GH :: FD: DE. | 4. |
| Also | 6 | $\angle A = \angle C$, $\angle G = \angle D$, $\angle ABG = \angle CFD$, $\angle AHG = \angle CED$. Const. | |
| ∴ e. 6. 5. 4. 2. | 7 | $\angle A = \angle C$, AB: AH :: CF: CE; $\angle G = \angle D$, BG: GH :: FD: DE; $\angle ABG = \angle CFD$, AB: BG :: CF: FD; $\angle AHG = \angle CED$, AH: GH :: CE: DE. | |
| ∴ e. 7. | 8 | ABGH, CFDE, are Similar. Def. 1. Q. E. F. & D. | |

LEMMA

LEMMA I.

If three Quantities A, AR, ARR , be continually Proportional, that is, if $A:AR::AR:ARR$; then as the First is to the Third, so is the Square of the First to the Square of the Second.

That is, $A:ARR::AA:AAAR$.

DEMONSTRATION.

- 1 $A \times ARR = AR \times AR$.
 2 $A:AR::AR:ARR$. Lem. 2. 5.
 Also 3 $A \times AARR = ARR \times AA$.
 4 $A:ARR::AA:AAAR$. Lem. 2. 5. Q. E. D.

PROP. XIX.

Similar Triangles ABC, DEF , are in the Duplicate Ratio, or as the Squares of their Homologous Sides BC, EF .

PREPARATION.

To the Sides BC, EF , find (11.) a third Proportional BG , so that $BC:EF::EF:BG$, and join AG .

I say, $\triangle ABC:\triangle DEF::BCq:EFq$.

DEMONSTRATION.



- 1 $AB:BC::DE:EF$. Def. 1.
 2 $BC:EF::AB:DE$. Cor. 7. 5.
 Also 3 $BC:EF::EF:BG$. Const.
 4 $AB:DE::EF:BG$. 11. 5.
 Again 5 $\angle B = \angle E$. Def. 1.
 6 $\angle B = \angle E, AB:DE::EF:BG$.
 7 $\triangle ABG = \triangle DEF$. 15.
 8 $BC:BG::\triangle ABC:\triangle ABG$. 1.
 9 $BC:BG::\triangle ABC:\triangle DEF$. Sub.
 Also e. 3. 10 $BC:BG::BCq:EFq$. Lem. 1.
 11 $\triangle ABC:\triangle DEF::BCq:EFq$. 11. 5. Q. E. D.

B b b

COROL.

COROLLARIES.

1 Because (Step. 9.) $BC:BG::\triangle ABC:\triangle DEF$, therefore if three Lines BC , EF , BG , be Proportional, then as the first BC , is to the third BG , so is the Triangle described upon the first BC , to the Similar Triangle described upon the second EF .

2 Because Equiangular Triangles are likewise (Scho. 1. 4.) Similar, therefore Equiangular Triangles, are also as the Squares of their Homologous Sides.

PROP. XX.

1st Similar Polygons $ABCDE$, $FGHIK$, are divided into an equal Number of Similar Triangles, ABC and FGH , ACD and FHI , ADE and FIK . 2d. Those Similar Triangles are Homologous to the wholes $ABCDE$, $FGHIK$, so that $\triangle ABC:\triangle FGH::ABCDE:FGHIK::\triangle ACD:\triangle FHI::\triangle ADE:\triangle FIK$. 3d. The Polygons $ABCDE$, $FGHIK$, are in the Duplicate Ratio, or as the Squares of their Homologous Sides CD , HI .

That is, 1st $\triangle ABC$ and FGH , ACD and FHI , ADE and FIK , are Similar.

DEMONSTRATION.



- | | |
|---|---|
| 1 | $\angle B = \angle G$, $BA:BC::GF:GH$. Def. 1. |
| 2 | $\triangle ABC$, FGH , are Equiangular. 6. |
| 3 | $CA:BC::HF:GH$. 4. |
| 4 | $\angle ACB = \angle FHG$. |
| 5 | $\angle ACB + \angle ACD = \angle FHG + \angle FHI$. Def. 1. |
| 6 | $\angle ACD = \angle FHI$. Ax. 12. |
| 7 | $BC:CD::GH:HI$. Def. 1. |
| 8 | $CA:BC::HF:GH$; $BC:CD::GH:HI$. |

CA

- $\therefore$ e. 8. 9 CA:CD::HF:HI. 22. 5.
 $\therefore$ e. 6. 9. 10 $\angle ACD = \angle FHI$, CA:CD::HF:HI.
 $\therefore$ e. 10. 11 $\triangle ACD$, FHI, are Equiangular. 6.
 Lastly 12 $\angle E = \angle K$, EA:ED::KF:KL. Def. 1.
 $\therefore$ e. 12. 13 $\triangle ADE$, FIK, are Equiangular. 6.
 $\therefore$ e. 11. 13. 14 $\triangle ABC$ and FGH, ACD and FHI, ADE and FIK, are Equiangular.
 $\therefore$ e. 14. 15 $\triangle ABC$ and FGH, ACD and FHI, ADE and FIK, are Similar. Scho. 1. 4. Q. E. D.

$$2d. \triangle ABC : \triangle FGH :: ABCDE : FGHIK :: \triangle ACD : \triangle FHI :: \triangle ADE : \triangle FIK.$$

DEMONSTRATION.

- 1 $\triangle ABC$ and FGH, ACD and FHI, are Similar. 1st.
 $\therefore$ e. 1. 2 $\triangle ABC : \triangle FGH :: CA : HF :: \triangle ACD : \triangle FHI$. 19.
 $\therefore$ e. 2. 3 $\triangle ABC : \triangle FGH :: \triangle ACD : \triangle FHI$. 11. 5.
 Again 4 $\triangle ACD$ and FHI, ADE and FIK, are Similar. 1st.
 $\therefore$ e. 4. 5 $\triangle ACD : \triangle FHI :: DA : IF :: \triangle ADE : \triangle FIK$. 1.
 $\therefore$ e. 5. 6 $\triangle ACD : \triangle FHI :: \triangle ADE : \triangle FIK$. 11. 5.
 $\therefore$ e. 3. 6. 7 $\triangle ABC : \triangle FGH :: \triangle ACD : \triangle FHI :: ADE : FIK$.
 $\therefore$ e. 7. 8 $\triangle ABC : \triangle FGH :: \triangle ABC + \triangle ACD + \triangle ADE : \triangle FGH + \triangle FHI + \triangle FIK$. 12. 5.
 i. e. 9 $\triangle ABC : \triangle FGH :: ABCDE : FGHIK$.
 $\therefore$ e. 9. 7. 10 $\triangle ABC : \triangle FGH :: ABCDE : FGHIK :: \triangle ACD : \triangle FHI :: \triangle ADE : \triangle FIK$. 11. 5. Q. E. D.

$$3d. ABCDE : FGHIK :: CDa : HIa.$$

DEMONSTRATION.

- 1 ABCDE:FGHIK:: $\triangle ACD$: $\triangle FHI$. 2d.
 And 2 CDa:HIa:: $\triangle ACD$: $\triangle FHI$. 19.
 $\therefore$ e. 1. 2. 3 ABCDE:FGHIK::CDa:HIa. 11. 5. Q. E. D.

COROLLARIES.

1 Similar Quadrilateral Figures ACDE, FHIK, are also in the duplicate Ratio, or as the Squares of their homologous Sides CD, HI.

For 1 $\triangle ACD : \triangle FHI :: \triangle ADE : \triangle FIK$. 2d
 ∴ c. 1. 2 $\triangle ACD : \triangle FHI :: \triangle ACD + \triangle ADE : \triangle FHI + \triangle FIK$. 12. 5.
 i. e. 3 $\triangle ACD : \triangle FHI :: ACDE : FHIK$.
 Also 4 $\triangle ACD : \triangle FHI :: CDq : HIq$. 19.
 ∴ c. 3. 4. 5 $ACDE : FHIK :: CDq : HIq$. 11. 5.

2 Therefore (19.) Similar Triangles, (Cor. 1.) Similar Quadrilateral Figures, (20.) Similar Polygons, and consequently universally, Similar Right-lined Figures, are in the duplicate Ratio, or as the Squares of their Homologous Sides.

3 If to CD, HI, be taken a third Proportional X, then as the first CD, is to the third X, so is the Quadrilateral Figure ACDE, described upon the first CD, to the Similar Quadrilateral Figure FHIK, described upon the second HI, so is the Polygon ABCDE, described upon the first CD, to the Similar Polygon FGHIK, described upon the second HI.

For 1 $CD : X :: CDq : HIq$. Lem. 1.
 Also 2 $ACDE : FHIK :: (Cor. 1.) CDq : HIq :: (3d.) ABCDE : FGHIK$.
 ∴ c. 1. 2. 3 $CD : X :: ACDE : FGHIK :: ABCDE : FGH$
 IK . 11. 5.

There

4

Therefore, if three Right-lines are Proportional, then as the First is to the Third, so (Cor. 1. 19.) is the Triangle, so (Cor. 3.) is the Quadrilateral Figure, and Polygon, and consequently so universally is the Right-lined Figure, described upon the First. to the Similar One described upon the Second.

P R O P. XXI.

Right-lined Figures ABC, DEI, which are Similar to the same Right-lined Figure HGF, are also Similar to one another.

That is, **ABC, DEI, are Similar.**

D E M O N S T R A T I O N.



- $\angle A = \angle H = \angle D$, $AB:AC::HG:HF::DE:DI$;
 $\angle C = \angle F = \angle I$, $AC:CB::HF:FG::DI:IE$; $\angle B =$
 $\angle G = \angle E$, $AB:CB::HG:FG::DE:IE$ Def. 1.
 $\therefore$ c. 1. 2 ABC, HGF, DEI , are Similar. Def. 1.
 $\therefore$ c. 2. 3 **ABC, DEI , are Similar. Q. E. D.**

C c c

PROP.

P R O P. XXII.

If four Right-lines AB, CD, EF, GH, are Proportional, that is, if $AB:CD::EF:GH$, the Similar Right-lined Figures KAB, LCD, MF, NH, described upon those Lines, are Proportional. And if the Similar Right-lined Figures KAB, LCD, MF, NH, are Proportional, that is, if $KAB:LCD::MF:NH$, the four Lines AB, CD, EF, GH, upon which they are described, are Proportional.

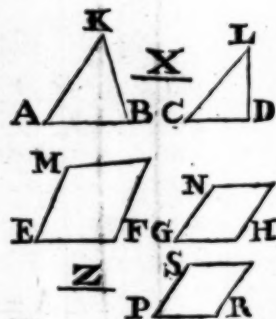
Hyp. 1 Suppose $AB:CD::EF:GH$.

P R E P A R A T I O N.

To AB, CD, find (11.) a third Proportional X, so that $AB:CD::CD:X$; also to EF, GH, find (11.) a third Proportional Z, so that $EF:GH::GH:Z$.

I say, ~~$KAB:LCD::MF:NH$~~ .

D E M O N S T R A T I O N.



- 1 $AB:CD::EF:GH$. *Hyp.*
 Also 2 $AB:CD::CD:X$. *Const.*
 ∴ e. 1. 2. 3 $EF:GH::CD:X$. 11. 5.
 Also 4 $EF:GH::GH:Z$. *Const.*
 ∴ e. 3. 4. 5 $CD:X::GH:Z$. 11. 5.
 ∴ e. 1. 5. 6 $AB:CD::EF:GH, CD:X::GH:Z$.
 ∴ e. 6. 7 $AB:X::EF:Z$. 12. 5.
 Also 8 $AB:X::KAB:LCD$. *Cor. 4. 20.*
 ∴ e. 7. 8. 9 $EF:Z::KAB:LCD$. 11. 5.
 Also 10 $EF:Z::MF:NH$. *Cor. 4. 20.*
 ∴ e. 9. 10. 11 ~~$KAB:LCD::MF:NH$~~ . 11. 5. Q. E. D.

Hyp. 2 Suppose $KAB:LCD::MF:NH$.

P R E.

PREPARATION.

To AB, CD, EF, find (12.) a fourth Proportional PR, so that $AB:CD::EF:PR$; then upon PR describe (18.) the Right-lined Figure SR, Similar to the Similar Right-lined Figures MF, NH.

I say, $AB:CD::EF:GH$.

DEMONSTRATION.

1 $AB:CD::EF:PR$. *Const.*

∴ e. 1. 2 $KAB:LCD::MF:SR$. *1st.*

Also 3 $KAB:LCD::MF:NH$. *Hyp.*

∴ e. 2. 3. 4 $MF:SR::MF:NH$. *11. 5.*

∴ e. 4. 5 $SR =$ to its Similar NH . *9. 5.*

∴ e. 5. 6 $PR = GH$.

∴ e. 1. 6. 7 $AB:CD::EF:GH$. *Sub. Q. E. D.*

PROP. XXIII.

Equiangular Parallelograms BD, BF, are in a Ratio compounded of that their Sides.

PREPARATION.

Let the Sides AB, BG, about the equal Angles, be set in a straight Line, so (Scho. 15. 1.) shall EBC, be likewise a straight Line; also let DC, FG, be produced till they meet in H.



I say, $\frac{\square BD}{\square BF} = \frac{AB}{BG} \times \frac{CB}{BE}$.

DEMONSTRATION.

1 $\square BD:\square BH::AB:BG$; $\square BH:\square BF::CB:BE$. *1.*

∴ e. 1. 2 $\frac{\square BD}{\square BH} = \frac{AB}{BG}$; $\frac{\square BH}{\square BF} = \frac{CB}{BE}$. *Def. 2. 5.*

Again 3 $\frac{\square BD}{\square BF} = \frac{\square BD}{\square BH} \times \frac{\square BH}{\square BF}$. *Def. 5.*

∴ e. 2. 3. 4 $\frac{\square BD}{\square BF} = \frac{AB}{BG} \times \frac{CB}{BE}$. *Sub. Q. E. D.*

C c c 2

PROP.

PROP. XXIV.

In every Parallelogram BD, the Parallelograms EG, KH, which are about the Diameter AC, are Similar to the whole BD, and also to one another.

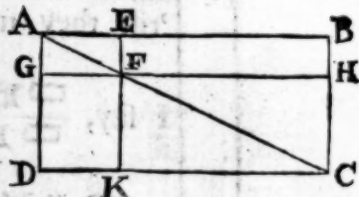
PREPARATION.

With the Right-lines GH, EK, drawn thro' the Point F, taken at Pleasure in the Diagonal AC, parallel to the opposite Sides, divide the Parallelogram BD, into four Parallelograms EG, EH, GK, KH, so (Scho. 2. 4.) are the two Triangles AEF, FHC, Similar to the Triangle ABC, the two AGF, FKC, Similar to the Triangle ADC, and consequently (21.) the three Triangles AEF, ABC, FHC, as also the three AGF, ADC, FKC, are Similar.

I say, $\square^{EG}, \square^{KH}, \square^{BD}$, are Similar.

DEMONSTRATION.

- | | | | |
|-----------------------|---|--|--------|
| | 1 | $\angle GFE = \angle GAE$. | 34. 1. |
| Also | 2 | $\angle GFE = \angle KFH$. | 15. 1. |
| $\therefore$ e. 1. 2. | 3 | $\angle GAE = \angle KFH$. | Ax. 1. |
| Also | 4 | $\angle KCH = \angle KFH$. | 34. 1. |
| $\therefore$ e. 2. 4. | 5 | $\angle GFE = \angle KCH$. | Ax. 1. |
| Again | 6 | $\angle AEF = \angle ABC = \angle FHC; \angle AGF = \angle ADC = \angle FKC$. | |
| | | 29. 1. | |
| $\therefore$ e. 6. | 7 | $AE:EF::AB:BC::FH:HC; AG:GF::AD:DC::FK:KC$. | |
| | | Def. 1. | |
| Now | 8 | $AE = GF, AB = DC, FH = KC$. | |
| $\therefore$ e. 7. 8. | 9 | $GF:EF::DC:BC::KC:HC; AG:AE::AD:AB::FK:FH$. | |
| | | Sub. | |



$\angle GFE$

...e. 9. 3. 9. 6. 7. 10

...e. 10. 11

$\angle GFB = \angle KCH$, $GF:EF::DC:BC::KC:HC$; $\angle GAE = \angle KPH$, $AG:AE::AD:AB::FK:FH$; $\angle AEF = \angle ABC = \angle FHC$, $AE:EF::AB:BC::FH:HC$; $\angle AGF = \angle ADC = \angle FKC$, $AG:GF::AD:DC::FK:KC$.

$\square ECG$, $\square BKH$, $\square BCD$, are Similar. Def. 1. Q. E. D.

PROP. XXV.

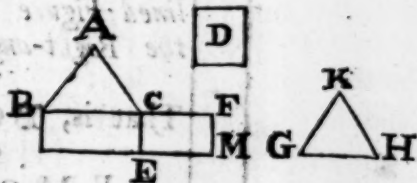
Unto a Right-lined Figure given ABC , to describe another KGH Similar, which shall also be equal to another Right-lined Figure given D .

PREPARATION.

Upon BC , make (44. 1.) $\square BE = ABC$, and upon CE , make (45. 1.) $\square CM = D$; betwixt BC and CE , find (13.) a mean Proportional GH , so that $BC:GH::GH:CE$, upon which make (18.) KGH Similar to ABC .

I say, $\square ECG = D$.

DEMONSTRATION.



$BC:CE::ABC:KGH$. Cor. 4. 20.

$BC:CE::\square BE:\square CM$. 1.

$ABC:KGH::\square BE:\square CM$. 11. 5.

$\square BE = ABC$. Const.

$ABC:KGH::ABC:\square CM$. Sub.

$KGH = \square CM$. 9. 5.

$D = \square CM$. Const.

$\square ECG = D$. Ax. 1. Q. E. F. & D.

We shall here omit the 26th, 27th, 28th, and 29th Propositions, that are but of little Consequence.

D d d

PROP.

PROP. XXX.

To cut a finite Right-line given AB, according to extreme and mean Ratio, so that $AB:AG::AG:GB$.

PREPARATION.

Make (II. 2.) $AGq = AB \times GB$.

I say, $AB:AG::AG:GB$.

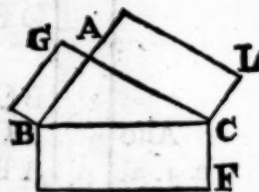
DEMONSTRATION.

I $AGq = AB \times GB$. Const. $A \text{---} \overset{G}{|} \text{---} B$
 ∴ c. 1. 2 $AB:AG::AG:GB$. 17. Q. E. F. & D.

PROP. XXXI.

In any Right-angled Triangle ABC, the Sum of any two Similar Right-lined Figures BG, AL, described upon the Sides AB, AC, is equal to the Similar Right-lined Figure BF, described upon the Side BC, subtending the Right-angle BAC.

That is, $BG + AL = BF$.



DEMONSTRATION.

1 $BG:BF::ABq:BCq$; $AL:BF::ACq:BCq$. Cor. 2. 20.
 ∴ c. 1. 2 $BG + AL:BF::ABq + ACq:BCq$. 24. 5.
 ∴ c. 2. 3 $BG + AL:ABq + ACq::BF:BCq$. 16. 5.
 ∴ c. 3. 4 $ABq + ACq:BG + AL::BCq:BF$. Cor. 7. 5.
 Now 5 $ABq + ACq = BCq$. 47. 1.
 ∴ c. 4. 5. 6 $BG + AL = BF$. 14. 5. Q. E. D.

The 32d, Proposition is useless.

PROP.

PROP. XXXIII.

In equal Circles ABC, DEF, the Angles have the same Ratio with their Arches BC, EF, on which they insist; whether they be at the Centers, as BGC, EHF, or at the Circumferences, as A, D.

PREPARATION.

Set off (the Circles being equal) Arch CI = Arch EF, and draw GI, so (27. 3.) is $\angle CGI = \angle EHF$.

I say, Arch BC : Arch EF :: $\angle BGC$: $\angle EHF$:: $\angle A$: $\angle D$.

DEMONSTRATION.



- 1 As the Measure of the $\angle BGC$: the Measure of the $\angle CGI$:: $\angle BGC$: $\angle CGI$.
 But 2 Arch BC is the Measure of the $\angle BGC$, Arch CI the Measure of the $\angle CGI$.
 ∴ e. 1. 2. 3 Arch BC : Arch CI :: $\angle BGC$: $\angle CGI$. Sub.
 Again 4 Arch CI = Arch EF, $\angle CGI = \angle EHF$. Const.
 ∴ e. 3. 4. 5 Arch BC : Arch EF :: $\angle BGC$: $\angle EHF$. Sub.
 Lastly 6 $\angle BGC : \frac{1}{2} \angle BGC :: \angle EHF : \frac{1}{2} \angle EHF$. 15. 5.
 ∴ e. 6. 7 $\angle BGC : \angle EHF :: \frac{1}{2} \angle BGC : \frac{1}{2} \angle EHF$. 16. 5.
 But 8 $\angle A = \frac{1}{2} \angle BGC$, $\angle D = \frac{1}{2} \angle EHF$. 20. 3.
 ∴ e. 7. 8. 9 $\angle BGC : \angle EHF :: \angle A : \angle D$. Sub.
 ∴ e. 5. 9. 10 Arch BC : Arch EF :: $\angle BGC$: $\angle EHF$:: $\angle A$: $\angle D$.
 11. 5. Q. E. D.

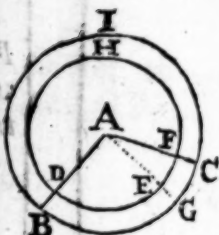
SCHOLIUM.

In the same Circle ABC, the Angles BGC, CGI, at the Center, (and consequently their Halves at the Circumference) have (Step. 3.) the same Ratio with their Arches BC, CI. The same is true of the Sectors GBCG, GCIG, as also of the Sectors GBCG, HEFH, in the same, and equal Circles, considering them as Angles.

CO.

CONOLLARIES.

Because as the Angle BAC , is to a Right-angle BAG , so (Scho. 33.) is the Arch BC to the Quadrant BG . Therefore as the Angle BAC , at the Center, is to four Right-angles, so is the Arch BC , on which it insists, to four Quadrants, that is, to the whole Periphery



The Arches DF , BC , of unequal Circles, which subtend equal Angles, whether at the Centers, as DAF , BAC , or at the Peripheries, are Similar; so that as the Arch DF , is to the whole Periphery DFH , so is the Arch BC , to the whole Periphery BCI .

- For 1. $\angle DAF : 4r \angle ::$ Arch $DF : DFH$. Cor. 1.
 Rec. 2. $\angle BAC : 4r \angle ::$ Arch $DF : DFH$.
 Also 3. $\angle BAC : 4r \angle ::$ Arch $BC : BCI$. Cor. 1.
 c. 2. 3. 4. Arch $DF : DFH ::$ Arch $BC : BCI$. II. 5.

Two Semidiameters AB , AC , cut off (Cor. 2.) Similar Arches DF , BC , from Concentric Peripheries.

The End of the Sixth Book.



EUCLID's ELEMENTS,

Demonstrated after a New, Plain, and Easie
METHOD.

BOOK XI.

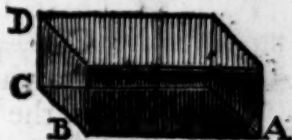
DEFINITIONS.

1



Body or **Solid**, is that which hath Length, Breadth, and Thickness.

As the Figure A D, whose Length is AB, Breadth BC, and Thickness CD.



Because a **Body** or **Solid**, hath Length, Breadth, and Thickness, it is called a Quantity of three Dimensions, and is supposed to consist of an Infinite Number of Surfaces.

2

The **Extremities**, or **Ends** of a Solid, are Surfaces or Surfaces.

E e e

A Right

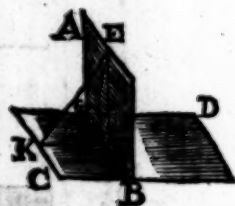
- 3 A Right-line is Perpendicular to a Plane, when 'tis Perpendicular to, or makes Right-angles with, all the Right-lines it meets in that Plane.

Thus the Right-line AB , is Perpendicular to the Plane CE , if it be Perpendicular to, or make Right-angles with, all the Right-lines BC , BD , BE , BF , it meets in that Plane.



- 4 One Plane is Perpendicular to another, when a Perpendicular Line drawn upon one of them to the common Section, is also Perpendicular to the other.

Thus the Plane AB , is Perpendicular to the Plane CD , because the Line EF , drawn upon one of the Planes AB , Perpendicular to the common Section GB , is also Perpendicular to the other CD .

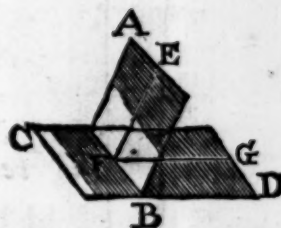


- 5 If the Line EK , be not Perpendicular to the Plane CD , and from the Point E , a Perpendicular EF , be drawn to it, and also the Line KF ; the Angle EKF , is the Inclination of the Line EK to the Plane CD .

- 6 The Inclination of one Plane to another, is the Acute Angle form'd by the two Perpendiculars, drawn upon each Plane, to their common Section.

Thus

Thus the **Inclination** of the Plane AB , to the Plane CD , is the Acute Angle EFG , form'd by the two Perpendiculars EF , GF , drawn upon each Plane to the common Section.

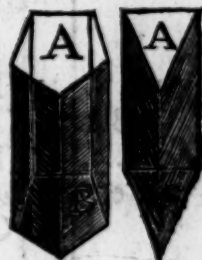


7 **Parallel Planes**, are such as being produced infinitely, will never meet.

8 A **Solid Angle** A , is that which is contained under more than two Plane Angles BAC , BAD , CAD , not being in the same Plane, but meeting all at one Point A .



9 A **Prism**, is a Solid Figure contained under Planes, whereof the two opposite A , B , are Parallel, Similar, and Equal; but the others are Parallelograms.



10 A **Parallelepipedon**, is a Solid Figure, contained under six Quadrilateral Figures, whereof those which are opposite are Parallel.

11 **Similar Solids**, are such as are bounded by an equal Number of Similar Planes.

12 **Similar and equal Solids**, are such as are bounded by an equal Number of Similar and equal Planes.

P R O P. I.

A Right-line CD, in a Plane AB, if produced, will still be in that Plane.

P R E P A R A T I O N.

Draw (11. 1.) from the Point D, in the Plane AB, the Right-line $DE \perp CD$, and another $DF \perp DE$.

I say, The Right-line CD produced, is in the Plane AB.

D E M O N S T R A T I O N.



- | | |
|-------------------------|--|
| 1 | $\angle EDC + \angle EDF = 2 \text{ r} \angle^\circ$. Def. 10. 1. |
| $\therefore$ e. 1. 2 | The Right-lines CD, DF, make one Right-line CF. 14. 1. |
| But 3 | The Right-lines CD, DF, are in the Plane AB. Const. |
| $\therefore$ e. 2. 3. 4 | The Right-line CF, is in the Plane AB. |
| i. e. 5 | The Right-line CD produced, is in the Plane AB. |
- Q. E. D.

P R O P. II.

Every Triangle ABC, is in the same Plane; as are also two Right-lines CD, BE, which cut each other.

P R E P A R A T I O N.

Draw at Pleasure in the $\triangle ABC$, the Right-line FG.

I say

I say, $\triangle ABC$ is in the same Plane,
as are also CD , BE .



DEMONSTRATION.

For 1 If part AFG , of the $\triangle ABC$, should be in one Plane,
part $BCGF$ in another.

Then 2 AF , FB , would be in different Planes.

But 3 AF , FB , are in the same Plane. 1.

∴ 4 Part of the $\triangle ABC$, is not in one Plane, part in another.

∴ 5 $\triangle ABC$ is in the same Plane.

∴ e. 5. 6 CA , BA , are in the same Plane.

∴ e. 6. 7 CD , BE , are in the same Plane. 1.

∴ e. 5. 7. 8 $\triangle ABC$ is in the same Plane, as are also CD , BE .
Q. E. D.

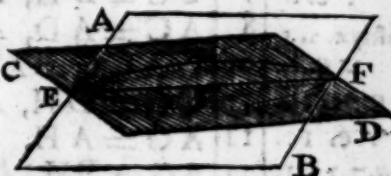
PROP. III.

The common Section EF , of two Planes AB , CD , is
a Right-line.

PREPARATION.

Thro' the Points E , F , of the common Section EF ,
draw (*post. 1.*) in the Plane AB , the Right-line EGF ,
in Plane CD , the Right-line EHF .

I say, The common Section
 EF is a Right-line.



DEMONSTRATION.

For 1 Either the Right-lines EGF , EHF , will coincide in the
common Section, and make one Right-line EF .

Or 2 Two Right-lines EGF , EHF , will include Space.

But 3 Two Right-lines cannot include Space. Ax. 20.

∴ 4 The Right-lines EGF , EHF , will coincide in the com-
mon Section, and make one Right-line EF .

∴ 5 The common Section EF is a Right-Line. Q. E. D.

F f f

PROP.

PROP. IV.

If a Right-line AB , be perpendicular to two other CD , EF , which cut each other in a Point B , so that ABC , ABD , ABE , ABF , be Right-angles, it will be also Perpendicular to the Plane of the same Lines.

PREPARATION.

Make $BC=BD=BE=BF$, and having joined EC , DF , and drawn thro' the Point B , the Line GH at Pleasure, join AE , AG , AC , AF , AH , AD .

I say, $AB \perp$ the Plane $ECFD$.

DEMONSTRATION.

- 1 $BC=BD=BE=BF$, AB is common,
 $\angle ABC=\angle ABD=\angle ABE=\angle ABF$. *Const.*
- 2 $AC=AD$, $AE=AF$. 4. 1.
- 3 $\angle CBE=\angle DBF$, $\angle CBG=\angle DBH$. 15. 1.
- 4 $BC=BD$, $BE=BF$, $\angle CBE=\angle DBF$.
- 5 $EC=DF$, $\angle BCG=\angle BDH$. 4. 1.
- 6 $BC=BD$, $\angle BCG=\angle BDH$, $\angle CBG=\angle DBH$.
- 7 $CG=DH$, $BG=BH$: 26. 1.
- 8 $AC=AD$, $AE=AF$, $EC=DF$.
- 9 $\angle ACG=\angle ADH$. 8. 1.
- 10 $CG=DH$, $AC=AD$, $\angle ACG=\angle ADH$.
- 11 $AG=AH$. 4. 1.
- 12 $BG=BH$, $AG=AH$, AB is common.
- 13 $\angle ABG=\angle ABH$. 8. 1.
- 14 $AB \perp GH$. *Def.* 10. 1.
- 15 After the same Manner, is $AB \perp$ all the Right-lines it meets in the Plane $ECFD$.
- 16 $AB \perp$ the Plane $ECFD$. *Def.* 3. Q. E. D.



PROP.

PROP. V.

If a Right-line AB, be perpendicular to three others BC, BD, BE, which cut each other at the same Point B, they will be all three in the same Plane.

PREPARATION.

If BE be said not to be the common Section of the Planes AE, FD, and consequently not to be in the same Plane with BC, BD, which (2.) are in the same Plane FD; let any other Right-line BG, be the common Section; so (AB being (Hyp.) $\perp$ BC, BD, and consequently (4.) to the Plane FD) will $\angle$ ABG, form'd by $\perp$ AB and that Right-line BG, be (Def. 3.) Right.

I say, BC, BD, BE, are in the same Plane

DEMONSTRATION.



For 1 If any other Line, besides BE, should be the common Section of the two Planes; as BG.

Then 2 $\angle$ ABG, form'd by $\perp$ AB and that Right-line BG would be Right.

Also 3 $\angle$ ABE is Right. Const.

$\therefore$ e. 3. 2. 4 $\angle$ ABE would be $= \angle$ ABG. Ax. 21.

But 5 $\angle$ ABE $\square$ $\angle$ ABG. Ax. 17.

$\therefore$ 6 No other Right-line, besides BE, is the common Section of the two Planes.

$\therefore$ 7 BE is the common Section of the two Planes.

$\therefore$ 8 BC, BD, BE, are in the same Plane. Q. E. D.

P R O P. VI.

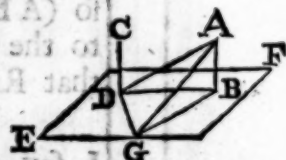
If two Lines AB , CD , be perpendicular to the same Plane EF , so that ABD , CDB , be Right-angles, those two Lines are parallel.

P R E P A R A T I O N.

In the Plane EF , draw (11. 1.) $GD \perp BD$ and $= AB$, so (Def. 10. 1.) is $\angle GDB$ Right, and consequently (Ax. 21.) $= \angle ABD$, draw also BG , AG , AD .

I say, $AB \parallel CD$.

D E M O N S T R A T I O N.



- | | | |
|---------------------------|----|---|
| | 1 | $GD = AB$, BD is common, $\angle GDB = \angle ABD$. <i>Const.</i> |
| $\therefore$ e. 1. | 2 | $AD = BG$. 4. 1. |
| Again | 3 | AG is common; $GD \perp BD$; ABD , CDB , are $r\angle$: <i>Const.</i> |
| $\therefore$ e. 1. 2. 3. | 4 | $GD = AB$, $AD = BG$, AG is common. |
| $\therefore$ e. 4. | 5 | $\angle GDA = \angle ABG$. 8. 1. |
| But | 6 | ABG is a $r\angle$. <i>Def. 3.</i> |
| $\therefore$ e. 5. 6. | 7 | GDA is a $r\angle$. |
| $\therefore$ e. 7. | 8 | $GD \perp AD$. <i>Def. 10. 1.</i> |
| Lastly | 9 | CDG is a $r\angle$. <i>Def. 3.</i> |
| $\therefore$ e. 9. | 10 | $GD \perp CD$. <i>Def. 10. 1.</i> |
| $\therefore$ e. 10. 8. 3. | 11 | $GD \perp CD$, AD , BD . |
| $\therefore$ e. 11. | 12 | CD , AD , BD , are in the same Plane. 5. |
| Also | 13 | AB , AD , BD , are in the same Plane. 2. |
| $\therefore$ e. 13. 12. | 14 | AB , CD , are in the same Plane. |
| $\therefore$ e. 14. 3. | 15 | AB , CD , are in the same Plane, ABD , CDB , are $r\angle$:. |
| $\therefore$ e. 15. | 16 | $AB \parallel CD$. 28. 1. Q. E. D. |

P R O P.

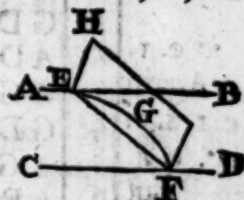
PROP. VII.

If there be two parallel Right-lines AB , CD , and any Points E , F , be taken in both of them, the Right-line EF , which joins the Points, is in the same Plane with those Parallels.

PREPARATION.

The Parallels AB , CD , being in one Plane, let the Right-line EF , be in another Plane HF , which cuts the Plane that AB , CD , are in, in the Points E , F , and draw BGF .

I say, EF is in the same Plane with AB , CD .

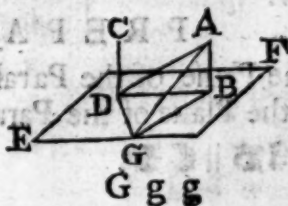


DEMONSTRATION.

- For 1 If any other Line, besides EF , should be the common Section of the two Planes; as BGF .
 Then 2 That Line EGF , would be a Right-line. 3.
 ∴ 3 Two Right-lines EF , EGF , would include Space.
 But 4 Two Right-lines cannot include Space. *Ax. 120.*
 ∴ 5 No other Line, besides EF , is the common Section of the two Planes.
 ∴ 6 EF is the common Section of the two Planes.
 ∴ 7 EF is in the same Plane with AB , CD . Q.E.D.

PROP. VIII.

If of two parallel Lines AB , CD , the one AB be perpendicular to a Plane EF , the other CD , will be also perpendicular to the same Plane.

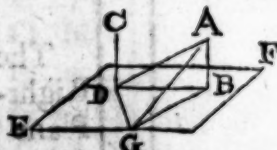


PRE-

P R E P A R A T I O N .

Draw the Line BD , cutting the Parallels in the Points B , D , so (ABD being (Def. 10. 1.) a Right-angle) will the Angle CDB , be (29. 1.) also Right. In the Plane EF , draw (11. 1.) $GD \perp BD$ and $= AB$, so (Def. 10. 1.) is $\angle GDB$ Right, and consequently (Ax. 21.) $= \angle ABD$, draw also BG , AG , AD .

I say, $CD \perp EF$.



D E M O N S T R A T I O N .

- 1 $GD = AB$, BD is common, $\angle GDB = \angle ABD$. *Const.*
 2 $\therefore$ e. 1. $AD = BG$. 4. 1.
 3 Again AG is common; GDB , CDB , are $r\angle$ s. *Const.*
 4 $\therefore$ e. 1. 2. 3. $GD = AB$, $AD = BG$, AG is common.
 5 $\therefore$ e. 4. $\angle GDA = \angle ABG$. 8. 1. *Q. E. D.*
 6 But ABG is a $r\angle$. *Def. 3.*
 7 $\therefore$ e. 5. 6. GDA is a $r\angle$.
 8 $\therefore$ e. 3. 7. GDB , GDA , are $r\angle$ s.
 9 $\therefore$ e. 8. $GD \perp BD$, AD . *Def. 10. 1.*
 10 $\therefore$ e. 9. $GD \perp$ the Plane of BD , AD . 4.
 11 But AB , CD , are in the same Plane with BD , AD . 7.
 12 $\therefore$ e. 10. 11. $GD \perp$ the Plane of AB , CD .
 13 $\therefore$ e. 12. CDG is a $r\angle$. *Def. 3.*
 14 $\therefore$ e. 3. 13. CDB , CDG , are $r\angle$ s.
 15 $\therefore$ e. 14. $CD \perp BD$, GD . *Def. 10. 1.*
 16 $\therefore$ e. 15. $CD \perp EF$. 4. *Q. E. D.*

P R O P . IX.

Right-lines AB , CD , which are parallel to the same Right-line EF , but not in the same Plane with it, are parallel one to the other.

P R E P A R A T I O N .

In the Plane of the Parallels AB , EF , draw $GH \perp EF$; also in the Plane of the Parallels CD , EF , draw $GI \perp EF$.

I say, $AB \parallel CD$.

D E M O N -

DEMONSTRATION.



- 1 $EG \perp GH, GI.$ *Const.*
 2 $EG \perp$ the Plane of $GH, GI.$ 4.
 Again 3 $AB \parallel EG.$ *Hyp.*
 4 $AB \parallel EG, EG \perp$ the Plane of $GH, GI.$
 5 $AB \perp$ the Plane of $GH, GI.$ 8.
 Lastly 6 $CD \parallel EG.$ *Hyp.*
 7 $CD \parallel EG, EG \perp$ the Plane of $GH, GI.$
 8 $CD \perp$ the Plane of $GH, GI.$ 8.
 9 $AB, CD,$ are $\perp$ the Plane of $GH, GI.$
 10 $AB \parallel CD.$ 6. Q. E. D.

PROP. X.

If two Right-lines $AB, AC,$ touching one another, be parallel to two other Right-lines $DE, DF,$ touching one another, but not in the same Plane with them, those Right-lines contain equal Angles $BAC, EDF.$

PREPARATION.

Make $AB = DE, AC = DF,$ and join $BC, EF, BE, AD, CF.$

I say, $\angle BAC = \angle EDF.$

DEMONSTRATION.



- 1 $AB \parallel DE, AC \parallel DF.$ *Hyp.*
 Also 2 $AB = DE, AC = DF.$ *Const.*
 3 $AB \parallel$ and $= DE, AC \parallel$ and $= DF.$
 4 $BE \parallel$ and $= AD, CF \parallel$ and $= AD.$ 33. 1.
 5 $BE = CF.$ *Ax. 1.*
 Also c. 4. 6 $BE \parallel CF.$ 9.
 7 $BE \parallel$ and $= CF.$
 8 $BC = EF.$ 33. 1.
 9 $AB = DE, AC = DF, BC = EF.$
 10 $\angle BAC = \angle EDF.$ 8. 1. Q. E. D.

G g g 2

PRO:
P

PROP. XI.

From a Point given A, without a Plane CD, to let fall a Line AB, perpendicular to the said Plane.

PREPARATION.

In the Plane CD, draw any Line EF; upon which, from the Point A, let fall (12. 1.) the $\perp$ AG; and from the Point G, in the Plane CD, draw (11. 1.) the $\perp$ GI, upon which, let fall (12. 1.) the $\perp$ AB, and thro' the Point B, draw (31. 1.) $HB \parallel EG$.

I say, $AB \perp CD$.

DEMONSTRATION.

- | | | |
|-----------------------|----|---|
| | 1 | $EG \perp AG$, GB. <i>Const.</i> |
| $\therefore$ e. 1. | 2 | $EG \perp$ the Plane AGB. 4. |
| Again | 3 | $HB \parallel EG$. <i>Const.</i> |
| $\therefore$ e. 3. 2. | 4 | $HB \parallel EG$, $EG \perp$ the Plane AGB. |
| $\therefore$ e. 4. | 5 | $HB \perp$ the Plane AGB. 8. |
| $\therefore$ e. 5. | 6 | ABH is a $r\angle$. <i>Def. 3.</i> |
| $\therefore$ e. 6. | 7 | $AB \perp HB$. <i>Def. 10. 1.</i> |
| Also | 8 | $AB \perp GB$. <i>Const.</i> |
| $\therefore$ e. 7. 8. | 9 | $AB \perp HB$, GB. |
| $\therefore$ e. 9. | 10 | $AB \perp CD$. 4. Q. E. D. |



PROP.

P R O P. XII.

From a Point given B, in a Plane EF, to erect a Right-Line AB, perpendicular to the said Plane.

P R E P A R A T I O N.

Upon the Plane EF, let fall (11.) any perpendicular CD, and thro' the Point B, draw (31. 1.) $AB \parallel CD$.

I say, $AB \perp EF$.

D E M O N S T R A T I O N.



1 $AB \parallel CD, CD \perp EF$. *Const.*
 2 $AB \perp EF$. 8. Q. E. D.

P R O P. XIII.

Two Right-lines, cannot be drawn perpendicular to a Plane, thro' the same Point.

P R E P A R A T I O N.

Thro' the Point C, in the Plane AB, draw the Right-lines CD, CE.

I say, CD and CE are not $\perp AB$.

D E M O N S T R A T I O N.



For 1 If CD and CE were $\perp AB$.
 Then 2 CD would be $\parallel CE$. 6.
 But 3 CD is not $\parallel CE$. *Def. 34. 1.*
 4 CD and CE are not $\perp AB$. Q. E. D.

H h h

P R O E

PROP. XIV.

Planes CD, EF, to which the same Right-line AB is Perpendicular, are parallel.

PREPARATION.

If CD, EF, be said not to be parallel, suppose them to be produced till they meet in some Line GH, in which taking any Point I, draw to it the Right-lines AI, BI, in the said Planes; so (Def. 3.) will IAB, IBA, be Right-angles.

I say, $CD \parallel EF$.

DEMONSTRATION.



- For 1 If CD, EF, being produced, should ever meet; as in GH.
 Then 2 IAB, IBA, two $\angle$'s of a Δ , would be $= 2r\angle$'s.
 But 3 Two $\angle$'s of a Δ are $\lhd 2r\angle$'s. 17. 1.
 $\therefore$ 4 CD, EF, being produced, will never meet; as in GH.
 $\therefore$ 5 $CD \parallel EF$. Def. 7. Q. E. D.

PROP. XV.

If two Lines AB, AC, touching one the other, be parallel to two others DE, DF, touching one the other, but not in the same Plane with them, their Planes BC, EF, are also parallel.

PREPARATION.

From the Point A, draw (11.) $AG \perp EF$, and thro' the Point G, where it meets the Plane, draw (31. 1.) in the same Plane, the Right-lines GH, GI, parallel to DE, DF, so (9.) are they also parallel to AB, AC; moreover HGA, IGA, are (Def. 3.) Right-angles.

I say,

I say, $BC \parallel EF$.

DEMONSTRATION.



- 1 $GH \parallel AB$, HGA is a $r\angle$. *Const.*
- 2 BAG is a $r\angle$. 29. 1.
- 3 $GI \parallel AC$, IGA is a $r\angle$. *Const.*
- 4 CAG is a $r\angle$. 29. 1.
- 5 BAG , CAG , are $r\angle$.
- 6 $AG \perp AB$, AC . *Def. 10. 1.*
- 7 $AG \perp BC$. 4.
- 8 $AG \perp EF$. *Const.*
- 9 $AG \perp BC$, EF .
- 10 $BC \parallel EF$. 14. Q. E. D.

PROP. XVI.

If two parallel Planes AB , CD , be cut by some other Plane EF , their common Sections are also parallel.

That is, $BC \parallel EF$.



DEMONSTRATION.

- 1 EH , GF , being produced, will still be in the Planes AB , CD , produced. 1.
- 2 The Planes AB , CD , being produced, will never meet. *Def. 7.*
- 3 EH , GF , being produced, will never meet.
- 4 $BC \parallel EF$. *Def. 34. 1.* Q. E. D.

PROP.

PROP. XVII.

If two Right-lines AB, CD , be cut by parallel Planes GH, IK, LM ; they shall be cut proportionally, so that $AE:EB::CF:FD$.

PREPARATION.

In the Planes GH, IK, LM , draw the Right-lines AC, EF, BD , as also AD , passing thro' the Plane IK , in the Point O .

I say, $AE:EB::CF:FD$.

DEMONSTRATION.

$EO \parallel BD$, 16.

$AE:EB::AO:OD$. 2. 6.

Also $OF \parallel AC$. 16.

$CF:FD::AO:OD$. 2. 6.

$AE:EB::CF:FD$. 11. 5. Q.E.D.



PROP. XVIII.

If a Right-line AB , be perpendicular to a Plane CD , all the Planes in which that Line is found, are perpendicular to the same Plane.

PREPARATION.

Thro' AB , draw any Plane EF , making the Section GF , with the Plane CD ; from some Point whereof I , draw (11. 1.) in the Plane EF , the $\perp HI$, so (ABI, HIB , being (Def. 10. 1.) Right-angles) is HI (28. 1.) $\parallel AB$.

I say

I say, All the Planes, in which AB is found, are $\perp$
CD.

DEMONSTRATION.



I $HI \parallel AB, AB \perp CD$. *Const.*

2. e. 1. | 2 | HI ⊥ CD. 8.

∴ e. 2. | 3 | $EF \perp CD$. Def. 4.

Also 4 After the same manner, are all the other Planes, in which
A B is found, | C D.

∴ 5 All the planes, in which AB is found, are $\perp$ CD.
Q. E. D.

PROP. XIX.

If two Planes AB, ED, which cut each other, be perpendicular to another Plane IK, their common Section EF, will be perpendicular to the same Plane.

PREPARATION.

From the Point F, draw (11. 1.) in the Planes AB, ED, the Right-lines FG, FH, perpendicular to the common Sections FB, FD, to (AB and ED being (*Hyp.*) $\perp$ IK) are FG and FH also (*Def. 4.*) $\perp$ IK.

I say, The common Section Of 1318 :



DEMONSTRATION.

For I Either $\perp$ FG, FH, will coincide in the
common Section, and make one E. F

Or 2 Two Right-lines can be drawn $\perp$ a Plane, thro' the same Point.

But 3 Two Right-lines cannot be drawn $\perp$ a Plane, thro' the same Point. 13.

∴ 4 L, FG, FH, will coincide in the common Section, and make one L EF.

5 The common Section of I J B. Q. E. D.

PROP. XX.

If a Solid Angle A, be contained under three Plane Angles BAC, BAD, CAD, any two of them howsoever taken, are greater than that which remains; so that the Sum of the Lesser Angles CAD, BAD, is greater than the greatest BAC.

PREPARATION.

From the greatest BAC, take (23. 1.) $\angle BAE = \angle BAD$, and making $AD = AE$, draw BEC, BD, DC.

I say, $\angle CAD + \angle BAD > \angle BAC$.

DEMONSTRATION.

- | | |
|----|---|
| 1 | $AD = AE$, AB is common, $\angle BAE = \angle BAD$. Const. |
| 2 | $BE = BD$. 4. 1. |
| 3 | $BD + DC < BC$. 20. 1. |
| 4 | $BE + DC < BC$. Sub. |
| 5 | $BE + DC < BE + EC$. |
| 6 | $DC < EC$. Ax. 16. |
| 7 | AC is common. |
| 8 | $AD = AE$, AC is common, $DC < EC$. |
| 9 | $\angle CAD < \angle CAE$. 25. 1. |
| 10 | $\angle CAD + \angle BAD < \angle CAE + \angle BAD$. Ax. 15. |
| 11 | $\angle CAD + \angle BAD < \angle CAE + \angle BAE$. Sub. |
| 12 | $\angle CAD + \angle BAD < \angle BAC$. Q. E. D. |



PROP.

PROP. XXI.

Every Solid Angle A, is contained under less Angles than four Right.

PREPARATION.

Having drawn a Plane, cutting the Sides of the Solid Angle any-wise, and making as many Triangles ABC, ACD, ADB, as the Plane hath Sides; denote all the Angles of the Plane by X, and the Sum of all the Angles at the Bases of the Triangles (every two of which, together with a Correspondent Angle of the Plane, form a Solid Angle) by Z.

I say, $\angle A < 4 \text{ right angles}$.

DEMONSTRATION.



1 $X =$ twice as many right angles , bating four, as the Plane hath Sides. The. 1. 1.

2 $X + 4 \text{ right angles} =$ twice as many right angles , as the Plane hath Sides.

Also 3 $Z + A =$ twice as many right angles , as the Plane hath Sides. 32. 1.

4 $Z + A = X + 4 \text{ right angles}$. Ax. 1.

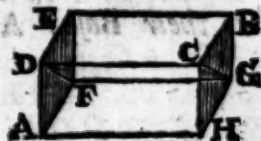
But 5 $Z < X$. 20.

6 $\angle A < 4 \text{ right angles}$. Q. E. D.

The 22d and 23d Propositions are needless.

PROP. XXIV.

If a Solid AB, be contained under parallel Planes, the opposite Planes thereof, are similar and equal Parallelograms.



PREPARATION.

Draw DF, CG.

I say, The opposite Planes are similar and equal $\square$.

DEMONSTRATION.

- 1 Each Plane cuts four others, whereof the Opposites are parallel.
- 2 The common Sections of each Plane with the opposite Planes are parallel. 16.
- 3 Each Plane is a $\square$. Def. 35. 1.
- 4 $AF = HG, AD = HC$. 34. 1.
- 5 $AF \parallel HG, AD \parallel HC$. Def. 35. 1.
- 6 $\angle DAF = \angle CHG$. 10.
- 7 $AF = HG, AD = HC, \angle DAF = \angle CHG$.
- 8 $\triangle DAF, CHG$, are equiangular and equal. 4. 1.
- 9 $\triangle DAF, CHG$, are similar and equal. Scho. 1. 4. 6.
- 10 $\square AE, HB$, their Doubles, are similar and equal.
- 11 After the same manner, are the other opposite Planes, similar and equal $\square$.
- 12 The opposite Planes are similar and equal $\square$. Q. E. D.



PROP. XXV.

If a Parallelepipedon AB, be cut by a Plane CD, parallel to the opposite Planes AF, EB; the Solids AC, EC, which are form'd by that Division, are to one another as their Bases AI, EI.

P R E P A R A T I O N.

Imagine Planes parallel to the common Base AG , to pass thro' all the Points of the Line AH , that may be taken for the common Height of the two Solids AC , EC ; and these Planes will divide the Solids into little Planes or Surfaces, equal in number in both, of which (*Def. 1.*) they are supposed to consist.

I say, $AG:EH::AC:EC$.



D E M O N S T R A T I O N.

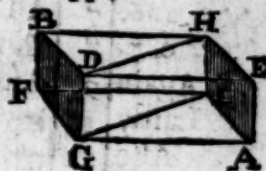
- 1 Each Plane of each Solid is = its Base. 24.
 ∴ e. 1. 2 $AI:EI::$ each Plane of AC : each Plane of EC .
 ∴ e. 2. 3 $AI:EI::$ all the Planes of AC : all the Planes of EC . 12. 5.
 i. e. 4 $AG:EH::AC:EC$. Q. E. D.

The 26th and 27th Propositions are needless.

P R O P. XXVIII.

A Parallelepipedon AB , is divided into two equal Prisms $BDHFGC$, $EDHAGC$, by a Plane DC , which passes thro' the two Diagonals GC , DH , of the opposite Planes FA , BE .

That is, $BDHFGC = EDHAGC$.



D E M O N S T R A T I O N.

- 1 DC is common. *Const.*
 And 2 BC and DA , BG and HA , are equal. 24.
 Also 3 BE and FA , are equal. 24.
 ∴ e. 3. 4 BDH and EDH , FGC and AGC , their halves, are equal.
 ∴ e. 1. 2. 4. 5 DC is common, BC and DA , BG and HA , BDH and EDH , FGC and AGC , are equal.
 ∴ e. 5. 6 $BDHFGC = EDHAGC$. *Def. 12.* Q. E. D.
 K k k PROP.

P R O P. XXIX, and XXX.

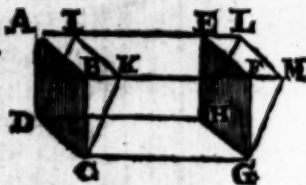
Solid Parallelepipedons CBADHEFG, CKIDHLMG, being constituted upon the same Base CDHG, and in the same Height, that is, between the same parallel Planes AM, DG, are equal one to the other.

Case 1

If the Parallelepipedons CBADHEFG, CKIDHLMG, have their insisting Lines CB, CK, GF, GM, DA, DL, HE, HL, placed within the same parallel Planes CBMG, DALH.

I say, $\square CBADHEFG = \square CKIDHLMG$.

DEMONSTRATION.



- | | | |
|-------------------------------|----|--|
| | 1 | DE, DL, are $\square$. 24. |
| $\therefore$ e. 1. | 2 | AD = EH, ID = LH. 34. 1. |
| Also e. 2. | 3 | AE = DH = IL. 34. 1. |
| $\therefore$ e. 3. | 4 | AE = IL. Ax. 1. |
| 4 - IE. | 5 | AE - IE = IL - IE. Ax. 12. |
| i. e. | 6 | AI = EL. |
| $\therefore$ e. 2. 6. | 7 | AD = EH, ID = LH, AI = EL. |
| $\therefore$ e. 7. | 8 | $\triangle ADI = \triangle EHL$. 8. 1. |
| Again | 9 | After the same Manner is $\triangle BCK = \triangle FGM$. |
| Lastly | 10 | $\square AK = \square EM$. 36. 1. |
| Also | 11 | $\square DB = \square HF$, $\square DK = \square HM$. 24. |
| $\therefore$ e. 8, 9, 10, 11. | 12 | $\triangle ADI = \triangle EHL$, $\triangle BCK = \triangle FGM$, $\square AK = \square EM$,
$\square DB = \square HF$, $\square DK = \square HM$. |
| $\therefore$ e. 12. | 13 | Prism ABCDIK = Prism EFGHLM. Def. 12. |
| $\therefore$ 13 + IDCKFEHG | 14 | ABCDIK + IDCKFEHG = EFGHLM + IDCKFEHG. Ax. 11. |
| i. e. | 15 | $\square CBADHEFG = \square CKIDHLMG$. Q. E. D. |

If

Case 2

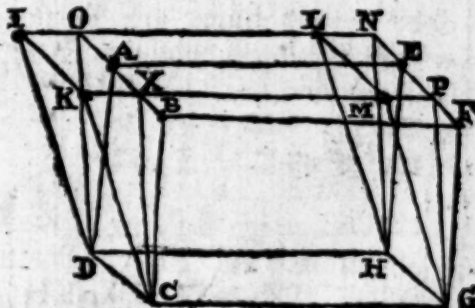
If the Parallelepipedons $CBADHEFG$, $CKIDHLMG$, have not their inscribing Lines CB , CK , GF , GM , DA , DI , HE , HL , placed within the same parallel Planes.

PREPARATION.

Having produced IL , FE , which being (*Hyp.*) in the same Plane will meet in the Point N , as also BA , KM , which being (*Hyp.*) in the same Plane will cut each other in the Point X , and meet IL , FE , (which (*Hyp.*) are in the same Plane with them) in the Points O , P , join CX , DO , HN , GP ; so will the opposite Quadrilateral Figures $PGHN$, $XCDO$, be parallel, they being in the same Planes with $FGHE$, $BCDA$, which are (*Def. 10.*) Parallel; also the opposite Quadrilateral Figures $ODHN$, $XCGP$, will be parallel, they being in the same Planes with $IDHL$, $KCGM$, which are (*Def. 10.*) Parallel; moreover the Quadrilateral Figure $XONP$, will be parallel to its opposite Quadrilateral Figure $CDHG$, it being in the same Plane with $BAEF$, which is (*Def. 10.*) parallel to $CDHG$, and consequently the Solid $CXODHNPG$, which has its opposite Sides parallel, will be (*Def. 10.*) a Parallelepipedon.

I say, $CBADHEFG = CKIDHLMG$.

DEMONSTRATION.



1 $CBADHEFG = CXODHNPG$. Case 1.

Alto 2 $CKIDHLMG = CXODHNPG$. Case 1.

∴ o. 1. 2. 3 $CBADHEFG = CKIDHLMG$. Ax. 1. Q.E.D. COROL.

K k k 2.

COROLLARY.

Triangular Prisms, which (18.) are half Parallelepipeds, being constituted on the same Base, and in the same Height, are (Ax. 8.) equal.

'Tis the same also in Polygonal Prisms, because they may be reduced into Triangular Prisms.

PROP. XXXI.

Solid Parallelepipeds EGH, AOH, being constituted upon equal Bases EG, AO, and in the same Height H, are equal one to the other.

Case 1

If the Parallelepipeds EGH, AOH, have the Sides perpendicular to the Bases.

PREPARATION.

To the Side FG produced, apply a Parallelogram GK, equal and similar to the Parallelogram AO, and having completed the Parallelogram GP, let the Right-lines PM, RG, meet KS, in Q and L, and imagine Parallelepipeds whose Sides are Perpendicular to their Bases, and whose common Height is H, to be constituted on GK, GQ, GP.

I say, $EGH = AOH$.



DEMON.

DEMONSTRATION.

- 1 EGH:GPH::EG:GP. 25.
 And 2 AO=EG. *Hyp.*
 ∴ e. 1. 2. 3 EGH:GPH::AO:GP. *Sub.*
 Again 4 GK=AO. *Const.*
 ∴ e. 3. 4. 5 EGH:GPH::GK:GP. *Sub.*
 And 6 GQ=GK. 35. 1.
 ∴ e. 5. 6. 7 EGH:GPH::GQ:GP. *Sub.*
 Also 8 GQH:GPH::GQ:GP. 25.
 ∴ e. 7. 8. 9 EGH:GPH::GQH:GPH. 11. 5.
 ∴ e. 9. 10 EGH=GQH. 9. 5.
 Also 11 GKH=GQH. *Case* 1. 29 and 30.
 ∴ e. 10. 11. 12 EGH=GKH. *Ax.* 1.
 Lastly 13 AO is equal and similar to GK. *Const.*
 ∴ e. 13. 14 AOH=GKH. *Def.* 12.
 ∴ e. 12. 14. 15 ~~EGH~~=~~AOH~~. *Ax.* 1. Q. E. D.

Case. 2 If the Parallelepipedons EGH, AOH, have not the Sides perpendicular to the Bases.

PREPARATION.

On the equal Bases EG, AO, and in the same Height, place the Parallelepipedons EGB, AOC, whose Sides are perpendicular to the Bases.

I say, ~~EGH~~=~~AOH~~.



DEMONSTRATION.



- 1 $EGB = EGH$. Case 1. 29 and 30.
 Also 2 $EGB = AOC$. Case 1.
 ∴ e. 1. 2. 3 $EGH = AOC$. Ax. 1.
 Also 4 $AOH = AOC$. Case 1. 29 and 30.
 ∴ e. 3. 4. 5 $EGB = AOC$. Ax. 1. Q. E. D.

COROLLARY.

Triangular Prisms, which (28.) are half Parallelepipeds, being constituted on equal Bases, and in the same Height, are (Ax. 8.) equal.

'Tis the same also in Polygonal Prisms, because they may be reduced into Triangular Prisms.

PROP. XXXII.

All Parallelepipeds of the same Height, are one to the other, as their Bases.

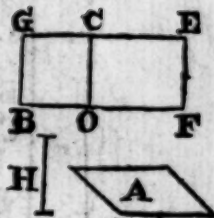
PREPARATION.

Let their Bases be A and BC, and their common Height H; so may the Parallelepipeds be expressed by AH and BCH. Upon CO make (44. 1.) the Parallelogram OE equal to the Parallelogram A, having the Angle OCE equal to the given Angle G. Upon OE, BC, imagine Parallelepipeds OEH, BCH, to be placed, of the same Height H; so will they be Parts of the one Parallelepipedon BEH.

I say,

I say, $AEH:BCH::A:BC$.

DEMONSTRATION.



1 $OEH:BCH::OE:BC$. 25.

And 2 $OE = A$. *Const.*

∴ c. 1. 2. 3 $OEH:BCH::A:BC$. *Sub.*

Again c. 2. 4 $OEH = AH$. 31.

∴ c. 3. 4. 5 $AEH:BCH::A:BC$. *Sub. Q. E. D.*

COROLLARY.

Triangular Prisms, which (28.) are half Parallelepipedons, of the same Height, are to one another as their Bases.

'Tis the same also in Polygonal Prisms, because they may be reduced into Triangular Prisms.

LEMMA I.

If four Quantities $A, AR, ARR, ARRR$, be continually proportional, that is, if $A:AR::AR:AR^2::AR^2:AR^3$; then as the First is to the Fourth, so is the Cube of the First, to the Cube of the Second.

That is, $A:AR^3::A^3:AR^3$.

DEMONSTRATION.

1 $A:AR::ARR:ARRR$. *Def. 2. 5.*

And 2 $A \times A^3 R^3 = A^3 \times AR^3$.

∴ c. 2. 3 $A:AR^3::A^3:AR^3$. *Lem. 2. 5. Q. E. D.*

L 112

LEM.

L E M M A II.

If four Quantities $A, AR, ARR, ARRR$, or AR, A, BR, B , either continued or discontinued, be proportional, their Cubes are also proportional.

That is, $A^3 : A^3 R^3 :: A^3 R^6 : A^3 R^9$, and $A^3 R^3 : A^3 :: B^3 R^3 : B^3$.

D E M O N S T R A T I O N.

And $\therefore$ e. 2. $\begin{array}{l} 1 \text{ } A : AR :: ARR : ARRR. \text{ and } AR : A :: BR : B. \text{ Def. 2. 5.} \\ 2 \text{ } A^3 \times A^3 R^3 = A^3 R^3 \times A^3 R^6. \text{ and } A^3 R^3 \times B^3 = A^3 \times B^3 R^3. \\ 3 \text{ } A^3 : A^3 R^3 :: A^3 R^6 : A^3 R^9. \text{ and } A^3 R^3 : A^3 :: B^3 R^3 : B^3. \\ \text{Lem. 2. 5. Q. E. D.} \end{array}$

S C H O L I U M.

Because (Lem. 2. 5.) $A^3 : A^3 R^3 :: A^3 R^6 : A^3 R^9$, and $A^3 R^3 : A^3 :: B^3 R^3 : B^3$; therefore if four Quantities, either continued or discontinued, be proportional, their Squares are also proportional.

P R O P. XXXIII.

Similar Solid Parallelepipedons EB, EN , are in the Triplicate Ratio, or as the Cubes, of their Homologous Sides EA, EF .

P R E P A R A T I O N.

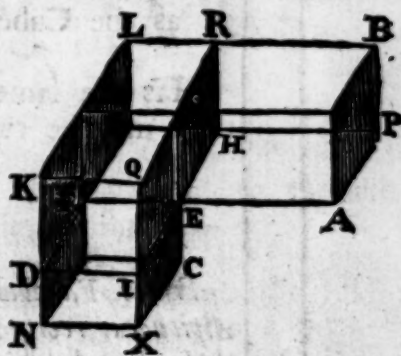
The Parallelepipedons EB, EN , being (Hyp.) similar, all their Planes will (Def. 12.) be similar, and consequently (Def. 1. 6.) $EA : EF :: EH : EI$, and $EH : EI :: EG : EC$. Also the Angles of their Planes will (Def. 1. 6.) be equal. Let therefore the Parallelepipedons EB, EN , be so placed, that the equal Angles HEA, FEI , as also the equal Angles

gles GEH, IEC, may be vertically Opposite in the same Planes; so (Scho. 15. 1.) will FEA, IEH, GEC, be straight Lines. Imagine upon the Planes EO, ED, Solids EL, EK, so placed, as that the Solids EB, EL, may make one Parallelepipedon AL; the Solids EL, EK, another Parallelepipedon IL; and the Solids EK, EN, a third Parallelepipedon GN.

I say, $EA:EF::EB:EN$.

DEMONSTRATION.

- | | | |
|-----------------|-----------------------|-----------------|
| 1 | $EA:EF::EP:EO$ | 1. 6. |
| Alfo 2 | $EB:EL::EP:EO$ | 25. |
| ∴ c. 1. 2. 3 | $EA:EF::EB:EL$ | 11. 5. |
| Alfo 4 | $EA:EF::EH:EI$ | Def. 1. 6. |
| ∴ c. 3. 4. 5 | $EB:EL::EH:EI$ | 11. 5. |
| Alfo 6 | $EO:ED::EH:EI$ | 1. 6. |
| ∴ c. 5. 6. 7 | $EB:EL::EO:ED$ | 11. 5. |
| Alfo 8 | $EL:EK::EO:ED$ | 25. |
| ∴ c. 7. 8. 9 | $EB:EL::EL:EK$ | 11. 5. |
| Again 10 | $EH:EI::ER:EQ$ | 1. 6. |
| Alfo 11 | $EL:EK::ER:EQ$ | 25. |
| ∴ c. 10. 11. 12 | $EH:EI::EL:EK$ | 11. 5. |
| Alfo 13 | $EH:EI::EG:EC$ | Def. 1. 6. |
| ∴ c. 12. 13. 14 | $EL:EK::EG:EC$ | 11. 5. |
| Alfo 15 | $EQ:EX::EG:EC$ | 1. 6. |
| ∴ c. 14. 15. 16 | $EL:EK::EQ:EX$ | 11. 5. |
| Alfo 17 | $EK:EN::EQ:EX$ | 25. |
| ∴ c. 16. 17. 18 | $EL:EK::EK:EN$ | 11. 5. |
| ∴ c. 9. 18. 19 | $EB:EL::EL:EK::EK:EN$ | 11. 5. |
| ∴ c. 19. 20 | $EB:EN::EB:EL$ | Lem. 1. |
| Alfo c. 3. 21 | $EA:EF::EB:EL$ | Lem. 2. |
| ∴ c. 21. 20. 22 | $EA:EF::EB:EN$ | 11. 5. Q. E. D. |



M m m

COROL.

COROLLARIES.

1 Because in continual Proportionals, as the First is to the Fourth, so (*Lem. 1.*) is the Cube of the First to the Cube of the Second; therefore if four Lines be continually Proportional, as the First is to the Fourth, so is a Parallelepipedon describ'd on the First, to a Similar Parallelepipedon describ'd on the Second.

2 Similar Triangular Prisms, which (*28.*) are the Halves of Similar Parallelepipedons, are in the Triplicate Ratio, or as the Cubes, of their Homologous Sides.

'Tis the same also in Similar Polygonal Prisms, because they may be reduced into Triangular Prisms.

PROP. XXXIV.

Equal Parallelepipedons AB, CD, have their Bases and Altitudes reciprocal, so that $AE:CF::CH:AG$. And those Parallelepipedons AB, CD, which have their Bases and Altitudes reciprocal, are equal.

Hyp. 1 Suppose $AB = CD$.

Case. 1 If the Parallelepipedons AB, CD, have their Sides perpendicular to their Bases, and their Altitudes CH, AG, equal

PREPARATION.

CH being (*Hyp.*) = AG, and $AB = CD$, the Ratio of CH to AG, is the same with that of AB to CD, and consequently (*Def. 2. 5.*) $CH:AG::AB:CD$.

I say,

I say, $AE:CF::CH:AG$.

DEMONSTRATION.



- 1 $AG=CH$. *Hyp.*
 $\therefore$ e. 1. 2 $AE:CF::AB:CD$. 32.
 Also 3 $CH:AG::AB:CD$. *Def. 2. 5.*
 $\therefore$ e. 2. 3. 4 $AE:CF::CH:AG$. 11. 5. Q. E. D.

Case 2 If the Parallelepipedons AB, CD, have their Sides perpendicular to their Bases, and their Altitudes CH, AG, unequal.

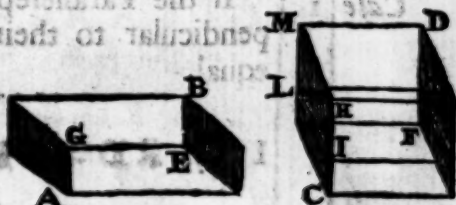
PREPARATION.

From the Greater CH, take $CI=AG$; and through I, draw the Plane $IK \parallel CF$.

I say, $AE:CF::CH:AG$.

DEMONSTRATION.

- 1 $AE:CF::AB:CK$. 32.
 And 2 $AB=CD$. *Hyp.*
 $\therefore$ e. 1. 2. 3 $AE:CF::CD:CK$. *Sub.*
 Also 4 $CM:CL::CD:CK$. 25.
 $\therefore$ e. 3. 4. 5 $AB:CF::CM:CL$. 11. 5.
 Also 6 $CH:CI::CM:CL$. 1. 6.
 $\therefore$ e. 5. 6. 7 $AE:CF::CH:CI$. 11. 5.
 Lastly 8 $CI=AG$. *Const.*
 $\therefore$ e. 7. 8. 9 $AE:CF::CH:AG$. *Sub.* Q. E. D.



Case 3

If the equal Parallelepipedons, have not their Sides perpendicular to their Bases, their Bases and Altitudes will be reciprocal.

DEMONSTRATION.

Imagine Parallelepipedons, which have their Sides perpendicular to their Bases, to be placed on the Bases of the Parallelepipedons, which have not their Sides perpendicular to their Bases, of the same Altitude with them; so (29 and 30) will they be equal to them. Since therefore the Parallelepipedons, which have their Sides perpendicular to their Bases, have (Case 1 and 2.) their Bases and Altitudes reciprocal, the Parallelepipedons which have not their Sides perpendicular to their Bases, but which are their Equals, will have also their Bases and Altitudes reciprocal. Q. E. D.

Hyp. 2

Suppose $AE:CF::CH:AG$.

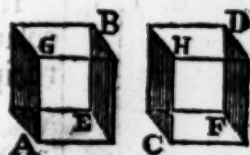
Case 1

If the Parallelepipedons AB, CD, have their Sides perpendicular to their Bases, and their Altitudes, CH, AG, equal.

I say, $AB = CD$.

DEMON-

DEMONSTRATION.



- 1 $AG = CH$. *Hyp.*
 ∴ e. 1. 2 $AE:CF::AB:CD$. 32.
 Also 3 $AE:CF::CH:AG$. *Hyp.*
 ∴ e. 2. 3. 4 $AB:CD::CH:AG$. 11. 5.
 ∴ e. 4. 5 $AB:CH::CD:AG$. 16. 5.
 ∴ e. 1. 5. 6 $AB:AG::CD:AG$. *Sub.*
 ∴ e. 6. 7 $AB = CD$. 9. 5. Q. E. D.

Case 2 If the Parallelepipedons AB , CD , have their Sides perpendicular to their Bases, and their Altitudes CH , AG , unequal.

PREPARATION.

From the greater CH , take $CI = AG$, and through I , draw the Plane $IK \parallel CF$.

I say, $AB = CD$.

DEMONSTRATION.

- 1 $AE:CF::CH:AG$. *Hyp.*
 And 2 $CI = AG$. *Const.*
 ∴ e. 1. 2. 3 $AE:CF::CH:CI$. *Sub.*
 Also 4 $AE:CF::AB:CK$. 32.
 ∴ e. 2. 4. 5 $CH:CI::AB:CK$. 11. 5.
 Also 6 $CH:CI::CM:CL$. 1. 6.
 ∴ e. 5. 6. 7 $AB:CK::CM:CL$. 11. 5.
 Also 8 $CD:CK::CM:CL$. 32.
 ∴ e. 7. 8. 9 $AB:CK::CD:CK$. 11. 5.
 ∴ e. 9. 10 $AB = CD$. 9. 5. Q. E. D.



Case 3

If the Parallelepipedons have not their Sides perpendicular to their Bases, and have their Bases and Altitudes reciprocal, they will be equal.

D E M O N S T R A T I O N .

Imagine Parallelepipedons, which have their Sides perpendicular to their Bases, to be placed on the Bases of the Parallelepipedons, which have not their Sides perpendicular to their Bases, of the same Altitude with them; so (29 and 30.) will they be equal to them. Since therefore the Parallelepipedons, which have their Sides perpendicular to their Bases, are (Case 1 and 2.) equal, the Parallelepipedons which have not their Sides perpendicular to their Bases, but which are their Equals, will be also equal. Q. E. D.

C O R O L L A R Y .

Equal Triangular Prisms, which (28.) are half Parallelepipedons, have their Bases and Altitudes reciprocal. And those Triangular Prisms, which have their Bases and Altitudes reciprocal, are equal.

'Tis the same also in Polygonal Prisms, because they may be reduced into Triangular Prisms.

The 35th Proposition is needless.

P R O P. XXXVI.

If three Right-lines AB, AC, AD, be continually proportional, that is, if $AB:AC::AC:AD$, the Parallelepipedon AE, made of these three Lines, is equal to a Parallelepipedon FH, which is Equiangular, and has all its Sides equal to the middle Line.

P R E.

PREPARATION.

From the Points C, L, draw the Lines CM, LN, perpendicular to the Planes of the Bases; so (the Solid Angles A, F, being supposed equal, so that if they could penetrate, neither would exceed the other, as also the Lines AC, FL) will the Heights CM, LN, be equal.

I say, $AC = FH$.

DEMONSTRATION.



- | | |
|---|--|
| 1 | $\angle BAD = \angle KFI$, $AB:AC::AC:AD$. <i>Hyp.</i> |
| 2 | $FK = AC = FL$. <i>Hyp.</i> |
| 3 | $\angle BAD = \angle KFI$, $AB:FK::FI:AD$. <i>Sub.</i> |
| 4 | $BD = KI$. 14. 6. |
| 5 | $BD:KI::AE:FH$. 32. |
| 6 | $BD:AE::KI:FH$. 16. 5. |
| 7 | $BD:AE::BD:FH$. <i>Sub.</i> |
| 8 | $AC = FH$. 9. 5. Q. E. D. |

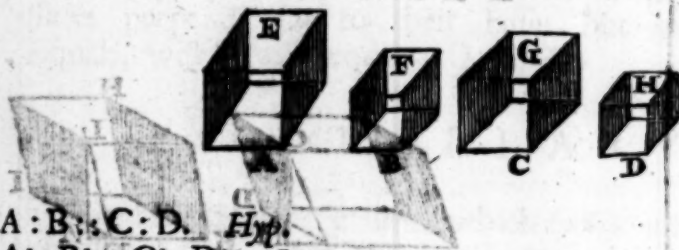
P R O P. XXXVII.

If four Lines A, B, C, D, be proportional, that is, if $A:B::C:D$, the Similar Parallelepipedons described upon those Lines are proportional. And if the Similar Parallelepipedons are proportional, those Lines shall be proportional.

Hyp. 1 Suppose $A:B::C:D$.

I say, $AE:BF::CG:DH$.

D E M O N S T R A T I O N.



1 $A:B::C:D$ Hyp.

∴ e. 1. 2 $A^3:B^3::C^3:D^3$ Lem. 2.

Also 3 $A^2:B^2::AE:BF$ 33.

∴ e. 2. 3. 4 $C^3:D^3::AE:BF$ 11. 5.

Also 5 $C^3:D^3::CG:DH$ 33.

∴ e. 4. 5. 6 $AE:BF::CG:DH$ 11. 5. Q. E. D.

Hyp. 2 Suppose $AE:BF::CG:DH$.

I say, $A:B::C:D$.

D E M O N S T R A T I O N.

1 $AE:BF::A^3:B^3$ 33.

Also 2 $AE:BF::CG:DH$ Hyp.

∴ e. 1. 2. 3 $A^3:B^3::CG:DH$ 11. 5.

Also 4 $C^3:D^3::CG:DH$ 33.

∴ e. 3. 4. 5 $A^3:B^3::C^3:D^3$ 11. 5.

∴ e. 5. 6 $A:B::C:D$ Q. E. D.

P R O P. XXXVIII.

If a Plane ED, be perpendicular to a Plane AC, and a Perpendicular EF, be drawn from a Point E, in one of the Planes ED, to the other Plane AC, that perpendicular EF, will fall on the common Section of the Planes AD.

P R E P A R A T I O N.

If EF be said to fall without AD, then in the Plane AC, let fall (12. 1.) $FG \perp AD$, and join EG; so (Def. 3.) will EFG, EGF, be Right-angles.

I say, EF will fall on AD.

D E M O N S T R A T I O N.



- For 1 If EF should fall without AD.
 Then 2 EFG, EGF, two $\angle^s$ of a Δ , would be $= 2 r\angle^s$.
 But 3 Two $\angle^s$ of a Δ are $\sqsubset 2 r\angle^s$. 17. 1.
 $\therefore$ 4 EF will not fall without AD.
 $\therefore$ 5 EF will fall on AD. Q. E. D.

I omit the 39th Proposition because of no great Consequence.

Q O O

PROP.

PROP. XL.

A Prism $ABCDEF$, whose Base is a Parallelogram AF , double the Triangular Base GHK , of another Prism $GHKLMN$, of the same Altitude, is equal to that other Triangular Prism.

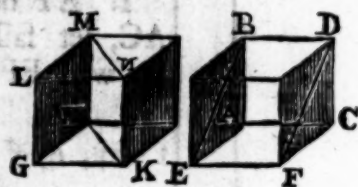
PREPARATION.

Compleat the Parallelepidons ED , HN .

I say, $ABCDEF = GHKLMN$.

DEMONSTRATION.

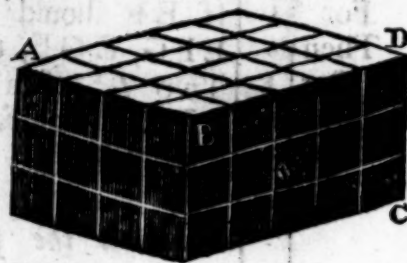
- | | | | |
|-----------------------|---|---|--|
| | 1 | $AF = 2\ GHK$. Hyp. | |
| Also | 2 | $HK = 2\ GHK$. 41. 1. | |
| $\therefore c. 1. 2.$ | 3 | $AF = HK$. Ax. 7. | |
| $\therefore c. 3.$ | 4 | $ED = HN$. 31. | |
| $4 \div 2$ | 5 | $\frac{1}{2}ED = \frac{1}{2}HN$. Ax. 14. | |
| Also | 6 | $\frac{1}{2}ED = ABCDEF$. 28. | |
| $\therefore c. 5. 6.$ | 7 | $\frac{1}{2}HN = ABCDEF$. Ax. 8. | |
| Also | 8 | $\frac{1}{2}HN = GHKLMN$. 28. | |
| $\therefore c. 7. 8.$ | 9 | $ABCDEF = GHKLMN$. Ax. 8. Q. E. D. | |



SCHOLIUM.

From the preceding Demonstrations, the Solidities, or Solid Contents of Parallelepipidons, as also of Triangular, and Mult-angular Prisms, are learnt; viz. by multiplying their Bases into their Altitudes.

As if the Altitude CD , of a Right-angled Parallelepipedon AC , be 3 Inches, its Length BD 5, and its Breadth AB 4; then (Lem. 1. 1.) is its Base $AD = AB \times BD = 4 \times 5 = 20$ Square Inches, which being multiplied by its



Altitude

Altitude CD , or 3 Inches, will produce 60 Cubick Inches, for the Solidity of the Right-angled Parallelepipedon $A C$.

For as a Rect-angle, so also is a Right-Parallelepipedon $A C$, produced of the Base $A D$, drawn into the Altitude CD . Therefore every Parallelepipedon is produced of the Base drawn into the Altitude, since (31.) every Parallelepipedon is equal to a Right-angled one, constituted upon the same Base, and in the same Altitude.

Moreover, since the whole Parallelepipedon is produced of the Altitude drawn into the Base, the half thereof (that is, (28.) a Triangular Prism) shall be produced of the Altitude drawn into half the Base.

'Tis the same also in Mult-angular Prisms, because they may be reduced into Triangular Prisms.

The End of the Eleventh Book.



EUCLID'S ELEMENTS,

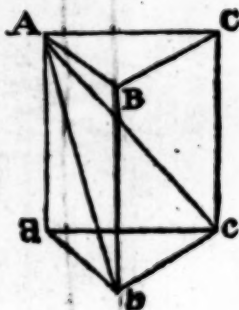
Demonstrated after a New, Plain, and Easie
METHOD.

BOOK XII.

DEFINITIONS.



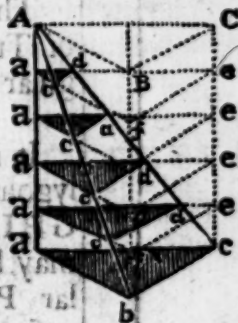
IF any Triangle ABC , be conceiv'd to move from the Vertex or Top of a Solid Angle, (determin'd by two Planes Aab , Aac , joined together in the common Line Aa) with a Motion always parallel to its self; so that its extreame Angular Point A , shall always remain in the Line Aa , but with its Sides AB , AC , shall all along raze on the Triangular Planes Aab , Aac , till at length they fall wholly within the Solid Angle, by this its Motion it will describe within the Solid Angle, the Figure called a **Pyramid**, whose Base will be the Triangular abc , and its Vertex A , and it



will

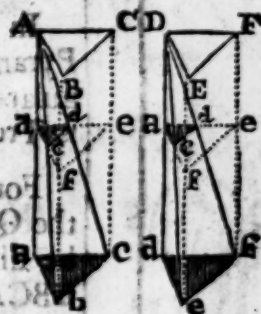
will also describe without it, another Quadrangular Pyramid, whose common Vertex will be the same A, but the Base the Quadrangle C b.

The first Pyramid it will describe with its Triangular Parts a c d, continually increasing from the Point A, and ending in the Triangle a b c, but the latter Pyramid will be described by the remaining Parts, or Trapezia c d e f, continually decreasing downwards from the whole A B C, and at Length ending in the Right-Line b c; so that in the mean while, the said Triangle with its whole Space, describes the Triangular Prism, compos'd of those two Pyramids.



COROLLARY I.

Of what Sort soever the Describing Triangles A B C, D E F, are, so they are equal; and whatever the Solid Angles are, comprehended under the Planes A a b, A a c, and D d e, D d f, so they are accommodated to the Plane Angles A and D, and such that the Describing Triangles A B C, D E F, conceiv'd to be mov'd within them, thro' an equal Interval of Motion from the Line A C to a c, and from D F to d f, may be all (or wholly) receiv'd within them in the same Moment of Time; because the increasing Triangular Planes a c d, in each Triangular Pyramid, as also the decreasing Trapezia c d e f, in each Quadrangular Pyramid, must necessarily be conceiv'd to answer one the other, both in Number and Magnitude, therefore all their Sums on both Sides will be equal, and consequently there will be described the equal Triangular Pyramids a b c A, d e f D, within the Solid Angles, and the equal Quadrangular Ones b c C B A, e f f E D, without the Solid Angles; and so Triangular and Quadrangular Pyramids, of equal Bases and Altitudes, may be undoubtedly taken for equal ones.

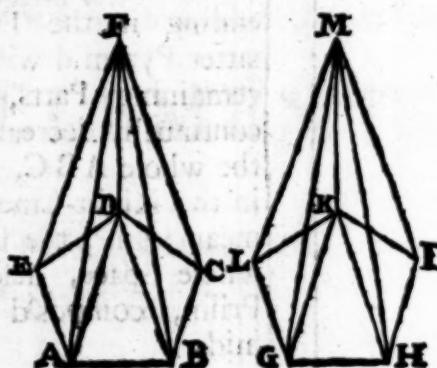


COROLLARY II.

Therefore all Pyramids, of equal Bases and Altitudes, are equal the one to the other.

This (*Cor. 1.*) is evident of Triangular and Quadrangular Pyramids.

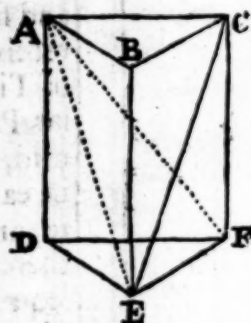
It is also as certain of Polygonal Pyramids $ABCDEF$, $GHIKLM$, because they may be resolved into Triangular Pyramids; which will answer one another both in Number and Magnitude, in each Polygonal Pyramid.



COROLLARY III.

In the Triangular Prism $ABCDEF$, the Quadrangular Pyramid $BCFEA$, is precisely Double of the other Triangular Pyramid $DEFA$; and the Triangular Prism $ABCDEF$, is Treble of the Triangular Pyramid $DEFA$.

For having drawn the Diagonal EC , the Quadrangular Pyramid $BCFEA$, will be divided into two Triangular Pyramids $EBCA$, $EFCA$, which will have a common Vertex A , and equal Triangular Bases, and consequently (*Cor. 2.*) will be equal.

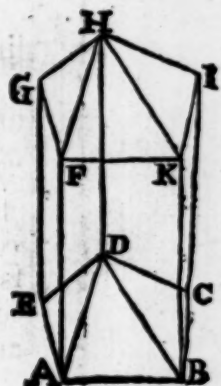


But one of these $EBCA$, or $ABCE$, (*viz.* if we take ABC for the Base, and E for the Vertex,) is (*Cor. 1.*) equal to $DEFA$. Therefore the Quadrangular Pyramid $BCFEA$, is $= 2 DEFA$, and consequently the Triangular Prism $ABCDEF$, is $= 3 DEFA$.
CO-

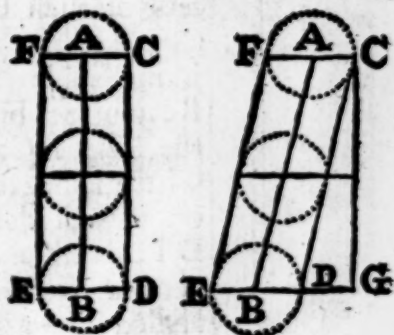
COROLLARY IV.

Hence, every Prism is Treble of the Pyramid, that has the same Base and Height with it ; or every Pyramid is the third Part of the Prism, that has the same Base and Height with it.

For resolve the Polygonal Prism $ABCDEGFKIH$, into Triangular Prisms $BCDHKI$, $BD AFKH$, $ADEGFH$; and the Pyramid $ABCDEG$, into Triangular Pyramids $BCDG$, $BDAG$, $ADEG$; so (Cor. 3.) will $BCDHKI$ be $= 3 BCDG$, $BD AFKH = 3 BDAG$, and $ADEGFH = 3 ADEG$, that is, all the Parts of the Prism will be Treble to all the Parts of the Pyramid, and consequently the whole Prism $ABCDEGFKIH$, is Treble the whole Pyramid $ABCDEG$.



- 2 If a Circle A , be conceiv'd to move downwards, along a Right-line CD , with a Motion always parallel to it self, by this its Motion it will generate a Solid $CDEF$, called a **Cylinder**, which, if the Right-line CD be perpendicular, is called a **Right-Cylinder**, if oblique, an **Inclined-Cylinder**.



COROLLARY I.

From this Genesis of Cylinders it is evident, that their Solid Contents may be found, by multiplying their Bases into their perpendicular Altitudes; because they must necessarily be conceiv'd to consist of as many Elements, or Circles, parallel and equal to the Describing Planes, or Bases A , as there are Points, or equal Parts, indefinitely many and small, in their perpendicular Altitudes AB , CG .

COROLLARY II.

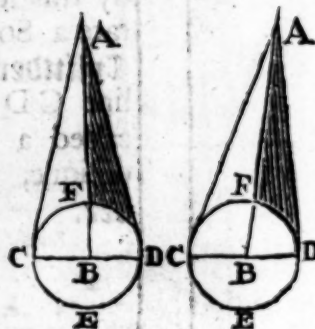
The Solidities therefore, both of Right and Inclined Cylinders, being found by multiplying their Bases A , into their perpendicular Altitudes AB , CG ; if their Bases A , and Altitudes AB , CG , be equal, their Solid Contents, or the Cylinders themselves, will be also equal.

Hence also it is evident, that Cylinders AB , AC , which are constituted upon the same Base A , and in the same Altitude AB , that is, within the same Parallels AD , BC , are equal.



The Axes of Cylinders, are those Right-lines AB , AC , which join the Centers of their Bases.

If a Triangle ABC , be conceiv'd to revolve about the immovable Side AB , till the Revolution be intire, that is, till the Side BC , arrive at the Place where it began; by this its Motion it will describe a Solid ACE DF , called a Cone, which, if generated by the Motion of a Right-angled Triangle, is called a Right-Cone, if by that of an Oblique angled Triangle, an Inclined Cone.



The

5 **The Axis of a Cone**, is that Right-line AB, which joins its Vertex A, and the Center B, of its Circular Base.

6 Right-Cylinders and Cones are **Similar**, when their Axes are proportional to the Diameters of their Bases ; but Inclined Cylinders and Cones are not **Similar**, unless they have a third Condition, viz. that their Axes be also equally inclined to the Planes of their Bases.

7 If a Semicircle ABC, be conceiv'd to move round the immoveable Diameter AC, till its Circumference ABC, come to the Place where it began to move, by this its Motion, it will describe a Solid ABCD, called a **Sphere**.



8 **The Axis of a Sphere**, is that immoveable Line AC, about which the Semicircle is turned.

9 **The Center of a Sphere**, is that Point from which all Right-lines drawn to the Surface, are equal ; as E.

10 **The Diameter of a Sphere**, is any Right-line whatsoever, which passes thro' its Center, and is bounded on each Side by its Surface.

PROPOSITION I.

Similar Polygonal Figures AB CDE, FGHIK, inscribed in Circles ABD, FGI, are one to another, as the Squares described of the Diameters of the Circles AL, FM.



P p p

P R E-

PREPARATION

Draw AC, BL, FH, GM.

I say, $ALq : FMq :: ABCDE : FGHIK$.

DEMONSTRATION.

- 1 $\angle ABC = \angle FGH$; $AB : BC :: FG : GH$. *Def. 1. 6.*
 $\therefore$ e. 1. 2 $\angle ACB = \angle FHG$. 6. 6.
 Also 3 $\angle ACB = \angle ALB$. 21. 3.
 $\therefore$ e. 2. 3. 4 $\angle FHG = \angle ALB$. *Ax. 1.*
 Also 5 $\angle FHG = \angle FMG$. 21. 3.
 $\therefore$ e. 4. 5. 6 $\angle ALB = \angle FMG$. *Ax. 1.*
 Again 7 $\angle ABL = \angle FGM$. 31. 3.
 $\therefore$ e. 6. 7. 8 $\angle ALB = \angle FMG$, $\angle ABL = \angle FGM$.
 $\therefore$ e. 8. 9 $\angle BAL = \angle GFM$. *Cor. 2. 32. 1.*
 $\therefore$ e. 8. 9. 10 $\angle ALB = \angle FMG$, $\angle ABL = \angle FGM$, $\angle BAL = \angle GFM$.
 $\therefore$ e. 10. 11 $AB : FG :: AL : FM$. *Scha. 3. 4. 6.*
 $\therefore$ e. 11. 12 $ABq : FGq :: ALq : FMq$. *Scho. Lem. 2. 11.*
 Also 13 $ABq : FGq :: ABCDE : FGHIK$. 20. 6.
 $\therefore$ e. 12. 13. 14 $ALq : FMq :: ABCDE : FGHIK$. 11. 5. Q.E.D.



LEMMA I.

The Peripheries of Circles are in Proportion one to another, as their Radius's.

PREPARATION.

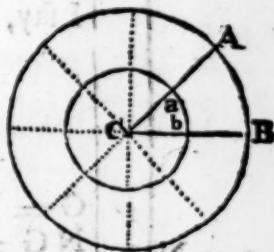
Having (by Radius's drawn from the Centers) divided the Periphery of a Circle, into any Number of equal Arches, suppose 8, whereof AB is one; thro' any Point in the Radius, draw a Concentric or Parallel Circle; so (*Cor.*

3. 33. 6.) will its Periphery be also divided into 8 equal Arches, by those Radius's, one of which will be $a b$. Moreover the Periphery of one Circle will be $8 a b$, that of the other $8 A B$, and ($\angle C$ being common, $\angle a$ (29. 1.) $= \angle A$, and $\angle b = \angle B$.) the Triangles $C a b$, $C A B$, will be Equiangular.

I say, $r : R :: p : P$.

DEMONSTRATION.

- $\therefore$ e. 1. 1 $C a : C A :: a b : A B$. Scho. 3. 4. 6.
 2 $C a : C A :: 8 a b : 8 A B$.
 i. e. 3 $r : R :: p : P$. Q. E. D.



COROLLARY.

Hence, if the Radius, or Semidiameter of one Circle, be called X , that of another Y , their Peripheries p , P , will have the same Proportion to one another, as $X Z$ to $Y Z$.

- For 1 $X : Y :: p : P$. Lem. 1.
 Also 2 $X : Y :: X Z : Y Z$. Lem. 2. 5.
 $\therefore$ e. 1. 2. 3 $p : P :: X Z : Y Z$. II. 5.

LEMMA II.

The Sum of a Series of Quantities, proceeding from 0 (inclusive) in an Arithmetical Progression, or whose common Difference is equal to the least Quantity; as 0, A , $2 A$, $3 A$, $4 A$, $5 A$, &c. is Subduple of the Product of the last or greatest Term, and Number of Terms.

P R E-

P R E P A R A T I O N.

Let the Sum of the Series be called S, the last or greatest Term G, and the Number of Terms N.

I say, $S = \frac{NG}{2}$.

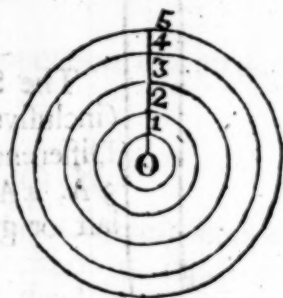
D E M O N S T R A T I O N.

$$\begin{array}{ll} 1 & G = N - 1 \times A. \\ 2 & NG = N - 1 \times NA. \text{ Ax. 13.} \\ 3 & \frac{NG}{2} = \frac{N - 1 \times NA}{2}. \text{ Ax. 14.} \end{array}$$

$$\begin{array}{ll} \text{Also } 4 & S = \frac{N - 1 \times NA}{2}. \\ 5 & S = \frac{NG}{2}. \text{ Ax. 1. Q. E. D.} \end{array}$$

C O R O L L A R I E S.

1 Hence, if a Circle be conceiv'd to consist of as many Concentric Peripheries, as there are Points, or equal Parts indefinitely many and small in the Radius; these Peripheries, as well as their Radius's, will (*Lem. 1.*) proceed from the Center, or o, in an Arithmetical Progression; and therefore their Sum, that is, the whole Circle, is equal to half the greatest (or extream Circumference) drawn into the Number of Terms, that is, the Radius.



Where-

2 Wherefore if the Radius, or Semidiameter of any Circle, be called X, that of another Y, their Peripheries or Circumferences will (Cor. Lem. 1.) be to one another, as XZ to YZ, and their Areas, as $\frac{1}{2} XZ X$ to $\frac{1}{2} YZ Y$.

PROP. II.

Circles are to one another, as the Squares of their Diameters.

PREPARATION.

Suppose the Radius, or Semidiameter of one Circle = X, that of another = Y; so will their Diameters be 2 X and 2 Y, and the Squares of their Diameters 4 X X and 4 Y Y; also the Area of one will (Cor. 2. Lem. 2.) be $\frac{1}{2} X Z X$, that of the other $\frac{1}{2} Y Z Y$.

I say, $4 X X : 4 Y Y :: \frac{1}{2} X Z X : \frac{1}{2} Y Z Y$.

DEMONSTRATION.

I. $4 X X \times \frac{1}{2} Y Z Y = 4 Y Y \times \frac{1}{2} X Z X$
 2. $4 X X : 4 Y Y :: \frac{1}{2} X Z X : \frac{1}{2} Y Z Y$. Lem. 2. 5. Q. E. D.

The third and fourth Propositions are needless; because they only serve to demonstrate the fifth and sixth, that I shall demonstrate otherwise, and more easily.

PROP.

PROP. V.

Triangular Pyramids AB, CD , of the same Height, are in Proportion one to another, as their Bases A, C .

PREPARATION.

Erect upon the Bases A, C , of the Triangular Pyramids AB, CD , Triangular Prisms of the same Height with them; so (Cor. 3. Def. 1.) will those Prisms be, among themselves, as $3 AB$ to $3 CD$.

I say, $A:C::AB:CD$.

DEMONSTRATION.

- 1 $A:C::3AB:3CD$. Cor. 32. 11.
 Alio 2 $AB:CD::3AB:3CD$. Lem.
 2. 5.
 3 $A:C::AB:CD$. 11. 5. Q. E. D.



PROP. VI.

All Sorts of Pyramids, of the same Height, have the same Proportion, as their Bases.

PREPARATION.

Let the Pyramids be $ABX, OFCZ$, and let their Bases AB, OFC , be resolved into the Triangles A, B, C, F, O , and the Pyramids themselves into the Triangular Pyramids AX, BX, CZ, FZ, OZ .

I say, $AB:OFC::ABX:OFCZ$.

DE-

DEMONSTRATION.

- 1 $AX:OZ::A:O; BX:OZ::B:O$ 5.
- ∴ e. 1. 2 $AX + BX:OZ::A + B:O$ 24.
- ∴ e. 2. 3 $AX + BX:A + B::OZ:O$ 16. 5.
- i. e. 4 $ABX:AB::OZ:O$
- Again 5 $AX:FZ::A:F; BX:FZ::B:F$ 5.
- ∴ e. 5. 6 $AX + BX:FZ::A + B:F$ 24. 5.
- ∴ e. 6. 7 $AX + BX:A + B::FZ:F$ 16. 5.
- i. e. 8 $ABX:AB::FZ:F$
- Lastly 9 $AX:CZ::A:C; BX:CZ::B:C$ 5.
- ∴ e. 9. 10 $AX + BX:CZ::A + B:C$ 24. 5.
- ∴ e. 10. 11 $AX + BX:A + B::CZ:C$ 16. 5.
- i. e. 12 $ABX:AB::CZ:C$
- ∴ e. 4. 8. 12. 13 $ABX:AB::OZ:O::FZ:F::CZ:C$ 11. 5.
- ∴ e. 13. 14 $ABX:AB::OZ + FZ + CZ:O + F + C$ 12. 5.
- ∴ e. 14. 15 $AB:O + F + C::ABX:OZ + FZ + CZ$ Cor. 7. 5.
- i. e. 16 $AB:OFC::ABX:OFCZ$ Q. E. D.



COROLLARIES.

- 1 Hence, All Sorts of Pyramids, of equal Bases and Altitudes, are equal.
- 2 Because (Cor. 4. Def. 1.) every Prism is Treble of the Pyramid, that has the same Base and Height with it; therefore all Sorts of Prisms of the same Height, have the same Proportion, as their Bases.

Q q q 2

PROP.

PROP. VII.

Every Prism is Treble of the Pyramid, that has the same Base and Height with it; or every Pyramid is the third Part of the Prism, that has the same Base and Height with it.

This is Demonstrated Cor. 4. Def. 1.

LEMMA III.

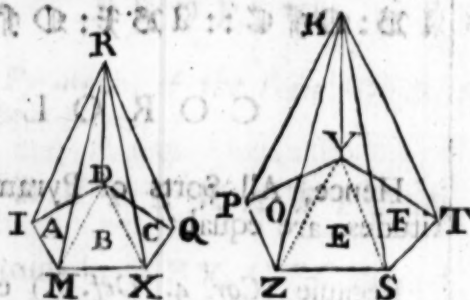
Similar Polygonal Pyramids $ABCR$, $OEFK$, may be resolved into the Similar Triangular Pyramids AR and OK , BR and EK , CR and FK .

PREPARATION.

Draw MD , XD , ZV , SV .

I say, AR and OK , BR and EK , CR and FK , are Similar.

DEMONSTRATION.



1. $\Delta^s A, O$, are Similar. 20. 6.
 2. $MD:MI::ZV:ZP$. Def. 1. 6.
 3. $\Delta^s MIR, ZPK$, are Similar. Def. 12. 11.

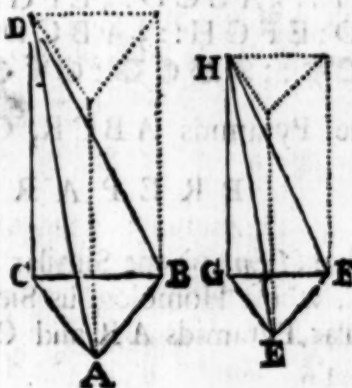
$MI:$

- $\therefore e. 3.$ 4 $MI:MR::ZP:ZK$. *Def. 1. 6.*
 $\therefore e. 2. 4.$ 5 $MD:MI::ZV:ZP$; $MI:MR::ZP:ZK$.
 $\therefore e. 5.$ 6 $MD:MR::ZV:ZK$. 22. 5.
 $\therefore e. 6.$ 7 $MR:MD::ZK:ZV$. *Cor. 7. 5.*
 Again 8 After the same manner, $MD:RD::ZV:KV$.
 $\therefore e. 7. 8.$ 9 $MR:MD::ZK:ZV$; $MD:RD::ZV:KV$.
 $\therefore e. 9.$ 10 $MR:RD::ZK:KV$. 22. 5.
 $\therefore e. 6. 8. 10.$ 11 $MD:MR::ZV:ZK$; $MD:RD::ZV:KV$; $MR:RD::ZK:KV$.
 $\therefore e. 11.$ 12 $\triangle MRD, ZKV$, are Equiangular. 5. 6.
 $\therefore e. 12.$ 13 $\triangle MRD, ZKV$, are Similar. *Scho. 1. 4. 6.*
 $\therefore e. 13.$ 14 AR, OK , are Similar. *Def. 12. 11.*
 Lastly 15 After the same manner, $\triangle XRD, SKV$, are Similar.
 $\therefore e. 15.$ 16 BR and EK , CR and FK , are Similar. *Def. 12. 11.*
 $\therefore e. 14. 16.$ 17 AR and OK , BR and EK , CR and FK , are Similar. Q. E. D.

P R O P. VIII.

Similar Pyramids are in the Triplicate Ratio, or as the Cubes of their Homologous Sides.

Case 1 If the Pyramids $ABCD, EFGH$, be Triangular.

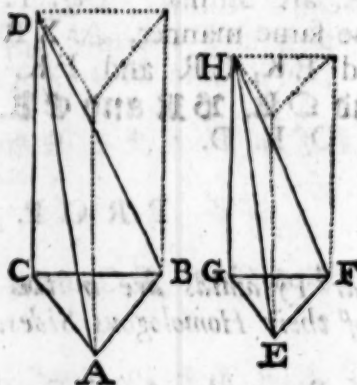


PREPARATION.

Upon the Similar Bases ABC , EFG , of the Similar Triangular Pyramids $ABCD$, $EFGH$, whose Homologous Sides are AB , EF , erect Similar Prisms of the same Heights with them; so (7.) will those Prisms be among themselves, as $3 ABCD$ to $3 EFGH$.

I say, $AB:EF::ABCD:EFGH$.

DEMONSTRATION.



Also
c. 1. 2.

- 1 $AB:EF::3 ABCD:3 EFGH$. Cor. 2. 33. 11.
- 2 $ABCD:EFGH::3 ABCD:3 EFGH$. Lem. 2. 5.
- 3 $AB:EF::ABCD:EFGH$. 11. 5. Q. E. D.

Case

- 2 If the Pyramids $ABCR$, $OEFK$, be Polygonal.

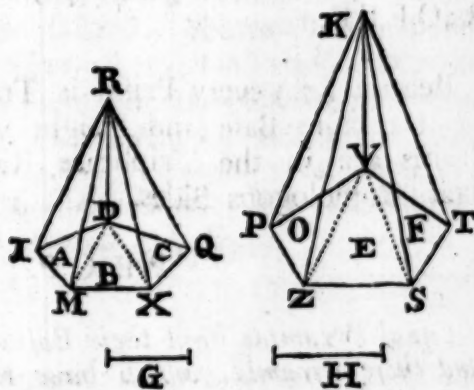
PREPARATION.

Resolve (Lem. 1.) the Similar Polygonal Pyramids $ABCR$, $OEFK$, whose Homologous Sides are IM , PZ , into Similar Triangular Pyramids AR and OK , BR and EK , CR and FK .

I say, $IM:PZ::ABCR:OEFK$.

DE.

DEMONSTRATION.



- 1 $IM:PZ::MX:ZS.$ Def. 1. 6.
 ∴ e. 1. 2 $I'M:P'Z::M'X:Z'S.$ Lem. 2. 11.
 Also 3 $BR:EK::M'X:Z'S.$ Case 1.
 ∴ e. 3. 3. 4 $I'M:P'Z::BR:EK.$ 11. 5.
 Again 5 $XQ:ST::MX:ZS.$ Def. 1. 6.
 ∴ e. 5. 6 $X'Q:S'T::M'X:Z'S.$ Lem. 2. 11.
 ∴ e. 2. 6. 7 $I'M:P'Z::X'Q:S'T.$ 11. 5.
 Also 8 $CR:FK::X'Q:S'T.$ Case 1.
 ∴ e. 7. 8. 9 $I'M:P'Z::CR:FK.$ 11. 5.
 Also 10 $I'M:P'Z::AR:OK.$ Case 1.
 ∴ e. 10. 4. 9. 11 $I'M:P'Z::AR:OK::BR:EK::CR:FK.$ 11. 5.
 ∴ e. 11. 12 $I'M:P'Z::AR+BR+CR:OK+EK+FK.$ 12. 5.
 i. e. 13 $I'M:P'Z::ABC:DEF.$ Q. E. D.

COROLLARIES.

1 Because in continual Proportionals, as the First is to the Fourth, to (Lem. 1. 11.) is the Cube of the First, to the Cube of the Second; therefore if four Lines be continually Proportional, as the First is to the Fourth, to is a Pyramid described on the First, to a Similar Pyramid described on the Second.

2 The Homologous Sides IM, PZ , of Similar Pyramids $ABCR, OEFK$, being therefore given; if IM, PZ, G, H , be continual Proportionals, it will be $IM:H::ABC R:OEFK$.

3 Because (7.) every Prism is Treble of the Pyramid, that has the same Base and Height with it; therefore Similar Prisms are in the Triplicate Ratio, or as the Cubes of their Homologous Sides.

PROP. IX.

Equal Pyramids have their Bases and Altitudes reciprocal: And those Pyramids which have their Bases and Altitudes reciprocal, are equal.

Hyp. 1 Suppose the Pyramids to be equal.

Case 1 If the Pyramids AB, CD , be Triangular.

PREPARATION.

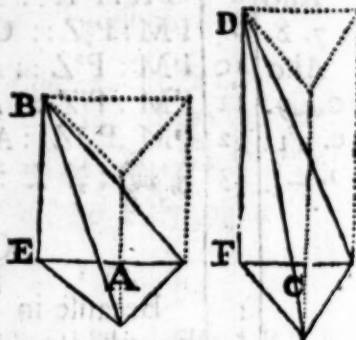
Erect upon their Bases A, C , Triangular Prisms, of the same Heights EB, FD , with them; so (they being (7.) Triple of the equal Pyramids) will those Prisms be also equal.

I say, $A:C::FD:EB$.

DEMONSTRATION.

1 $A:C::FD:EB$. Cor. 34. II. Q. E. D.

Case 2 If the equal Pyramids be Polygonal; their Bases and Altitudes will be reciprocal.



D E.

DEMONSTRATION.

Imagine the given Polygonal Bases, to be (25. 6.) resolved into Triangular Bases, equal to them; and upon those Triangular Bases, imagine Triangular Pyramids to be erected, of the same Heights with the given Polygonal Pyramids; so (Cor. 1. 6.) will they be equal to the equal Polygonal Pyramids, and consequently to one another. Since therefore those equal Triangular Pyramids, have (Case 1.) their Bases and Altitudes reciprocal, the given Polygonal Pyramids, their Equals, will also have their Bases and Altitudes reciprocal. Q. E. D.

Hyp. 2 Suppose the Pyramids to have their Bases and Altitudes reciprocal

Case 1 If the Pyramids AB, CD, be Triangular.

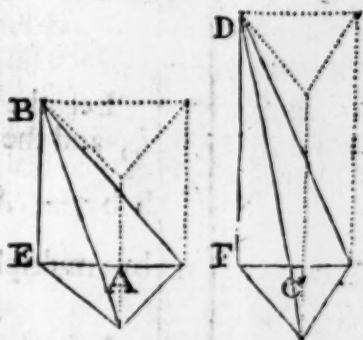
Erect upon their Bases A, C, Triangular Prisms, of the same Heights EB, FD, with them; so (7.) will those Prisms be among themselves, as 3 AB to 3 CD.

I say, $AB = CD$.

1 $3 AB = 3 CD$. Cor. 34. 11.

2 $AB = CD$. Ax. 14. Q. E. D.

Case 2 If the Pyramids be Polygonal; they will be equal.



DEMONSTRATION.

Imagine the given Polygonal Bases, to be (25. 6.) resolved into Triangular Bases, equal to them; and upon those Triangular Bases, imagine Triangular Pyramids to be erected, of the same Heights with the given Polygonal Pyramids; so (Cor. 1. 6.) will they be equal to the given Polygonal Pyramids, and have their Bases and Altitudes reciprocal. Since therefore those Triangular Pyramids are (Case 1.) equal, the given Polygonal Pyramids, their Equals, will be also equal. Q. E. D.

fff

COROL.

COROLLARY.

Because (7.) every Prism is Treble of the Pyramid, that has the same Base and Height with it; therefore equal Prisms have their Bases and Altitudes reciprocal: And those Prisms which have their Bases and Altitudes reciprocal, are equal.

LEMMA IV.

If the first Term of never so many continual Proportionals, A, AR, ARR, ARRR, &c. *ad Infinitum*, be subtracted from the last, and the Remainder divided by the Ratio lessened by Unity, the Quotient will be equal to the Sum of all the Terms except the last.

PREPARATION

Let the Number of Terms be called N, the last Term L, and the Sum of all the Terms except the last S; so will

S be $= A + AR + ARR = \frac{AR^N - A}{R - 1}$; as will appear by multiplying $R - 1$ into $A + AR + ARR$.

I say, $\frac{L - A}{R - 1} = S$.

DEMONSTRATION.

$$\begin{array}{ll}
 1 & L = AR^{N-1} \\
 2 & L - A = AR^{N-1} - A \quad \text{Ax. 12.} \\
 3 & \frac{L - A}{R - 1} = \frac{AR^{N-1} - A}{R - 1} \quad \text{Ax. 14.} \\
 \text{Also } 4 & S = \frac{AR^{N-1} - A}{R - 1} \\
 5 & \frac{L - A}{R - 1} = S \quad \text{Ax. 1. Q. E. D.}
 \end{array}$$

CO.

COROLLARIES.

1 In a continued Series of Fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \&c.$ decreasing in a Duple Proportion *ad Infinitum*, if you take them backwards, you may justly reckon a Cypher, or 0, for the first Term, (for between $\frac{1}{2}$ and 0, there may be an infinite Number of such Terms,) and the Sum of all the Terms except the last will be $= \frac{1}{2}$; for subtracting the first 0, from the last $\frac{1}{2}$, and dividing the Remainder $\frac{1}{2}$, by the Ratio lessened by Unity, that is, by 1, which divides Nothing, the Quotient $\frac{1}{2}$, is (*Lem. 4.*) the Sum of all the Terms, except the last.

2 The Sum of $\frac{1}{4} + \frac{1}{16} + \frac{1}{64}, \&c.$ decreasing in a Quadruple Proportion *ad Infinitum*, will be $\frac{1}{4}$; for subtracting the first Term 0, from the last $\frac{1}{4}$, and dividing the Remainder $\frac{1}{4}$, by the Ratio lessened by Unity, that is, by 3, the Quotient $\frac{1}{12}$ will (*Lem. 4.*) give the Sum of all the Terms except the last. This therefore (*viz.* $\frac{1}{4}$) being added, the Sum of all will be $= \frac{1}{4} = \frac{1}{4}$.

3 Wherefore $\frac{1}{2} - \frac{1}{4} = \frac{1}{4} - \frac{1}{8} = \frac{1}{8} - \frac{1}{16}, \&c. ad infinitum$, or $-\frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8}, \&c. ad Infinitum$, will (*Cor. 1.*) be $= 0$; also $\frac{1}{4} - \frac{1}{16} = \frac{1}{16} - \frac{1}{64} = \frac{1}{64} - \frac{1}{256}, \&c. ad Infinitum$, or $-\frac{1}{4} + \frac{1}{16} + \frac{1}{16} + \frac{1}{64}, \&c. ad Infinitum$, will (*Cor. 2.*) be $= 0$.

LEMMA V.

The Sum of a Series of Quantities, proceeding in a Duplicate Arithmetick Progression from 1, *ad Infinitum*, that is, 1, 4, 9, 16, $\&c.$ the Squares of Numbers in an Arithmetick Progression 1, 2, 3, 4, $\&c.$ is Subtriple of the Sum of as many Terms equal to the greatest, as is the Number of Terms.

DE.

DEMONSTRATION.

The Sum of three Terms, 1, 4, 9, = 14, is to thrice 9 = 27, (dividing both Sides by 9) as $1\frac{1}{3}$, or $1\frac{2}{3}$, or $1 + \frac{1}{3} + \frac{1}{3}$ to 3.

The Sum of Six Terms 1, 4, 9, 16, 25, 36, = 91, is to Six Times 36 = 216, (dividing both Sides by 72) as $1 + \frac{1}{4} + \frac{1}{9}$ to 3.

The Sum of twelve Terms 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, = 650, is to Twelve Times 144 = 1728 (dividing both Sides by 576) as $1 + \frac{1}{4} + \frac{1}{9}$ to 3.

The Fractions adhering to them thus constantly decreasing, some by their half Parts, others by three Quarters, (for $\frac{1}{16}$ is $\frac{1}{16}$, therefore the first Decrement is $\frac{1}{16}$; and $\frac{1}{16}$ is $\frac{1}{16}$, therefore the second Decrement is $\frac{1}{16}$;) the Sum of an infinite Series, will be to the Sum of as many Terms equal to the greatest,

$$1 + \frac{1}{4} + \frac{1}{16}$$

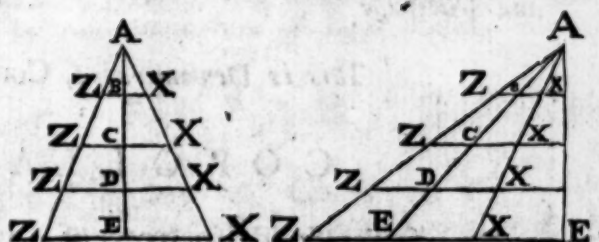
$$- \frac{1}{4} - \frac{1}{16}$$

$$- \frac{1}{16} - \frac{1}{64}$$

$$- \frac{1}{64}, \text{ \&c. } - \frac{1}{256}, \text{ \&c. to 3, that is, (Cor. 3. Lem. 4.)}$$

as 1 to 3. Q. E. D.

COROLLARY.



Hence, if we suppose the Axis AE , either of a Right, or Inclined Cone AZX , to be divided into equal and indefinitely many Parts, by as many parallel Right-lines ZX , the Lines ZX , will (it being (*Scho.* 3. 4. 6.) as $AB:AC::BZ:CZ::BX:CX$, and $AD:AE::DX:EX$.) be as the Numbers 1, 2, 3, 4, &c. and their Squares, or (2.) the Circles constituted on the Diameters ZX , as 1, 4, 9, 16, &c.

Whence all those Circles, or the whole Cone AZX , (made up of the same) will be Subtriple of as many Circles equal to the Greatest, constituted on the greatest Diameter ZEX , that is, Subtriple of a Cylinder whose Base is ZEX , and Axis or Altitude AE .

T t t

PROP.

PROP. X.

Every Cone is the third Part of a Cylinder of the same Base and Altitude.

This is Demonstrated, Cor. Lem. 5.

COROLLARY.

Because the Solid Contents of Cylinders, may (Cor. 1. Def. 2.) be learnt, by multiplying their Bases into their perpendicular Altitudes, and every Cone is the third Part of a Cylinder, of the same Base and Altitude; therefore the Solid Contents of Cones, may be learnt, by multiplying a third Part of their Bases into their perpendicular Altitudes.

PROP. XL

Cylinders and Cones, of the same Altitude, are as their Bases.

PREPARATION.

Let the Circular Bases, of the Cylinders and Cones, be (Cor. 2. Lem. 2.) expressed by $\frac{1}{3}XZX$, and $\frac{1}{3}YZY$, and their equal Altitudes by A; so (Cor. 1. Def. 2.) will the Cylinders be among themselves, as $\frac{1}{3}AXZX$ to $\frac{1}{3}AYZY$, and (Cor. 10.) the Cones, as $\frac{1}{9}AXZX$ to $\frac{1}{9}AYZY$.

I say, $\frac{1}{3}XZX : \frac{1}{3}YZY :: \frac{1}{9}AXZX : \frac{1}{9}AYZY :: \frac{1}{9}AXZX : \frac{1}{9}AYZY$.

DEMONSTRATION.

$$1 \quad \frac{1}{3}XZX : \frac{1}{3}YZY :: \frac{1}{9}AXZX : \frac{1}{9}AYZY. \text{ Lem. 2. 5.}$$

$$2 \quad \frac{1}{3}XZX : \frac{1}{3}YZY :: \frac{1}{9}AXZX : \frac{1}{9}AYZY. \text{ Lem. 2. 5.}$$

$$3 \quad \frac{1}{9}AXZX : \frac{1}{9}AYZY :: \frac{1}{9}AXZX : \frac{1}{9}AYZY. \text{ Q. E. D.}$$

Also

∴ c. 1. 2.

LEM

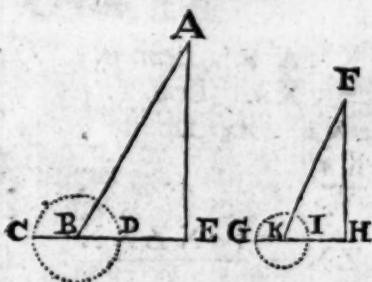
LEMMA VI.

Similar Cylinders and Cones, have their Altitudes proportional to the Diameters of their Bases.

PREPARATION.

If the Cylinders and Cones are Right, their Axes or Altitudes, will (*Def. 6.*) be proportional to the Diameters of their Bases; if Inclined, from the Points A, F, of their Axes, AB, FK, upon the Diameters of their Bases CD, GI, produced, let fall the Perpendiculars AE, FH, so ($\angle AEB$ being (*Ax. 21.*) $= \angle FHK$, $\angle ABE$ (*Def. 6.*) $= \angle FKH$) will $\angle BAE$ be (*Cor. 2. 32. 1.*) $= \angle KFH$ and consequently the Triangles ABE, FKH, will be Equiangular.

I say, $CD:GI::AE:FH$.



DEMONSTRATION.

I AB:FK::CD:GI. *Def. 6.*
 Also 2 AB:FK::AE:FH. *Scho. 3. 4. 6.*
 3 $CD:GI::AE:FH$. *II 5. Q. E. D.*
 T t t 2

PROP.

PROP. XII.

Similar Cylinders and Cones, are in the Triplicate Ratio, or as the Cubes of the Diameters of their Bases.

PREPARATION.

The Semi-diameters of the Circular Bases, of the Similar Cylinders and Cones, being (*Cor. 2. Lem. 2.*) expressed by X and Y , and their Bases by $\frac{1}{2}XZX$ and $\frac{1}{2}YZY$, let their Altitudes be represented by A and a ; so (*Cor. 1. Def. 2.*) will the Cylinders be among themselves as $\frac{1}{2}AXZX$ to $\frac{1}{2}aYZY$, and (*Cor. 10.*) the Cones as $\frac{1}{6}AXZX$ to $\frac{1}{6}aYZY$. Moreover the Diameters of their Bases being $2X$ and $2Y$, their Cubes, will be $8XXX$ and $8YYY$.

I say, $8X^3 : 8Y^3 :: \frac{1}{6}AXZX : \frac{1}{6}aYZY :: \frac{1}{6}AXZX : \frac{1}{6}aYZY$.

DEMONSTRATION.

- | | | |
|---------------------------|----|--|
| | 1 | $2X : 2Y :: A : a$. <i>Lem. 6.</i> |
| $\therefore e. 1.$ | 2 | $2YA = 2Xa$. <i>Lem. 1. 5.</i> |
| $2 \div 2$ | 3 | $YA = Xa$. <i>Ax. 14.</i> |
| $3 \div Y$ | 4 | $A = \frac{Xa}{Y}$. <i>Ax. 14.</i> |
| $4 \times \frac{1}{2}XZX$ | 5 | $\frac{1}{2}AXZX = \frac{\frac{1}{2}aXZXX}{Y}$. <i>Ax. 13.</i> |
| Again | 6 | $8X^3 : 8Y^3 :: \frac{\frac{1}{2}aXZXX}{Y} : \frac{1}{2}aYZY$. <i>Lem. 2. 5.</i> |
| $\therefore e. 5. 6.$ | 7 | $8X^3 : 8Y^3 :: \frac{1}{2}AXZX : \frac{1}{2}aYZY$. <i>Sub.</i> |
| Also | 8 | $\frac{1}{2}AXZX : \frac{1}{2}aYZY :: \frac{1}{6}AXZX : \frac{1}{6}aYZY$. <i>Lem. 2. 5.</i> |
| $\therefore e. 7. 8.$ | 9 | $8X^3 : 8Y^3 :: \frac{1}{6}AXZX : \frac{1}{6}aYZY :: \frac{1}{6}AXZX : \frac{1}{6}aYZY$ |
| | 11 | 5. Q. E. D. |

COROL

COROLLARIES.

1 Because the Axes of Similar Cylinders and Cones, are (Def. 6.) in the same Ratio, as the Diameters of their Bases; therefore Similar Cylinders and Cones, are in the Triplicate Ratio, or as the Cubes of their Axes.

2 Because the Altitudes of Similar Cylinders and Cones, are (Lem. 6.) in the same Ratio, as the Diameters of their Bases; therefore Similar Cylinders and Cones, are in the Triplicate Ratio, or as the Cubes of their Altitudes.

P R O P. XIII.

If a Cylinder A B, be cut by a Plane C, parallel to its Base A, the Cylinders made by that Section, will be to one another, as the Parts of the Axis A C, C B.

P R E P A R A T I O N

Having let fall the Right-line B E, perpendicular to the Plane of the Base A, which will be also perpendicular to the Plane of the Parallel Base C, let the Lines A E, C D, be made the common Sections of the two Parallel Planes A, C, and the Triangular Plane B A E; so (16. 11.) will the two common Sections A E, C D, be parallel. Let A, C, the equal Circular Bases of the Cylinders, be (Cor. 2. Lem. 2.) expressed by $\frac{1}{2} X Z X$, and their Altitudes E D, D B, by A and a; so (Cor. 1. Def. 2.) will the Cylinders be among themselves as $\frac{1}{2} A X Z X$ to $\frac{1}{2} a X Z X$.

I say, $AC:CB::\frac{1}{2}AXZX:\frac{1}{2}aXZX$.

D E M O N S T R A T I O N.

1 E D: D B:: A C: C B. 2. 6.

2 A: a:: A C: C B.

3 A: a:: $\frac{1}{2} A X Z X: \frac{1}{2} a X Z X$. Lem. 2. 5.

4 $AC:CB::\frac{1}{2}AXZX:\frac{1}{2}aXZX$. 11. 5.

Q. E. D.

U u u

If



i. e.
Also
e. 2. 3.

If the Cylinders made by the Section are Right, then their Axes are their Altitudes A , a , and consequently (*Lem.* 2. 5.) $A:a::\frac{1}{2}AXZX:\frac{1}{2}aXZX$.

P R O P. XIV.

Cylinders and Cones, which are constituted upon equal Bases, are to one another as their Altitudes.

P R E P A R A T I O N.

Let the equal Circular Bases of the Cylinders and Cones, be (*Cor.* 2. *Lem.* 2.) expressed by $\frac{1}{2}XZX$, and their Altitudes by A and a ; so (*Cor.* 1. *Def.* 2.) will the Cylinders be among themselves, as $\frac{1}{2}AXZX$ to $\frac{1}{2}aXZX$, and (*Cor.* 10.) the Cones as $\frac{1}{6}AXZX$ to $\frac{1}{6}aXZX$.

I say, $A:a::\frac{1}{6}AXZX:\frac{1}{6}aXZX::\frac{1}{2}AXZX:\frac{1}{2}aXZX$.

D E M O N S T R A T I O N.

$$1 \quad A:a::\frac{1}{2}AXZX:\frac{1}{2}aXZX. \text{ Lem. 2. 5.}$$

$$\text{Also } 2 \quad A:a::\frac{1}{6}AXZX:\frac{1}{6}aXZX. \text{ Lem. 2. 5.}$$

$$\therefore \text{e 1. 2. } 3 \quad A:a::\frac{1}{6}AXZX:\frac{1}{6}aXZX::\frac{1}{2}AXZX:\frac{1}{2}aXZX. \\ 11. 5. \quad Q. E. D.$$

P R O P.

PROP. XV.

Equal Cylinders BH, EK, and Cones BAC, EDI, have their Bases and Altitudes reciprocal; so that as $BC:EI::DF:AG$. And those Cylinders BH, EK, and Cones BAC, EDI, which have their Bases and Altitudes reciprocal, are equal.

Hyp. 1 Suppose $BH = EK$. $BAC = EDI$.

Case 1 If the Cylinders BH, EK, and Cones BAC, EDI, be Right, and have their Altitudes DF, AG, equal.

PREPARATION.

DF being (*Hyp.*) = AG, and $BH = EK$, the Ratio of DF to AG, is the same with that of BH to EK, and consequently (*Def. 2. 5.*) $DF:AG::BH:EK$.

I say, $BC:EI::DF:AG$.

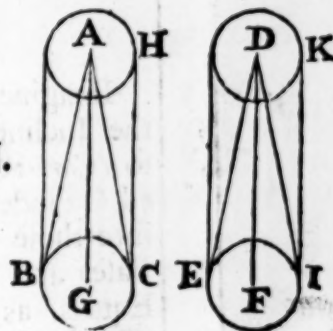
DEMONSTRATION.

1 $DF = AG$. *Hyp.*

2 $BC:EI::BH:EK$. 11.

3 $DF:AG::BH:EK$. *Def. 2. 5.*

4 $BC:EI::DF:AG$. 11. 5.



Q. E. D.

If

Case.

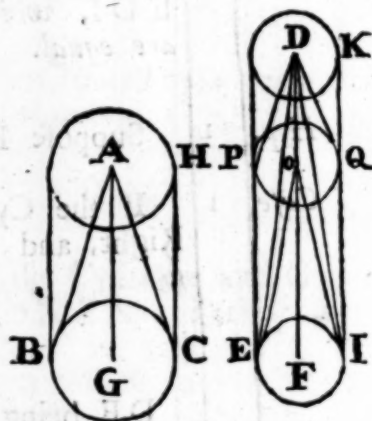
- 2 If the Cylinders BH, EK, and Cones BAC, EDI, be Right, and have their Altitudes DF, AG, unequal.

PREPARATION.

From the Greater DF, take $OF = AG$, and thro' O, draw $PQ \parallel EI$.

I say, $BC:EI::DF:AG$.

DEMONSTRATION.



- 1 $DF:OF::EK:EQ$ 14.
 And 2 $OF = AG$ *Const.*
 ∴ e. 1. 2. 3 $DF:AG::EK:EQ$ *Sub.*
 Again 4 $BH = EK$ *Hyp.*
 ∴ e. 3. 4. 5 $DF:AG::BH:EQ$ *Sub.*
 Also 6 $BC:EI::BH:EQ$ 11.
 ∴ e. 6. 5. 7 $BC:EI::DF:AG$ 11. 5. Q. E. D.

Case

- 3 If the equal Cylinders and Cones be Inclined; their Bases and Altitudes will be reciprocal.

DEMONSTRATION.

Imagine Right Cylinders to be placed on the Bases of the Inclined Cylinders, of the same Heights with them; so (*Cor. 2. Def. 2.*) will they be equal to the equal Inclined Cylinders, and consequently to one another. Since therefore those equal Right-Cylinders, have (*Case 1. and 2.*) their Bases and Altitudes reciprocal, the Inclined Cylinders, their Equals, as also the Inclined Cones, (which (*10.*) are the Thirds of the equal Inclined Cylinders, of the same Bases and Altitudes,) will have their Bases and Altitudes reciprocal.

Sup-

Hyp. 2 Suppose $BC:EI::DF:AG$.

Case 1 If the Cylinders BH , EK , and Cones BAC , EDI , be Right, and have their Altitudes DF , AG , equal.

I say, $BH = EK$. $BAC = EDI$.

DEMONSTRATION.

1 $DF = AG$. *Hyp.*

2 $BC:EI::BH:EK$. 11.

3 $BC:EI::DF:AG$. *Hyp.*

4 $BH:EK::DF:AG$. 11. 5.

5 $BH:DF::EK:AG$. 16. 5.

6 $BH:AG::EK:AG$. *Sub.*

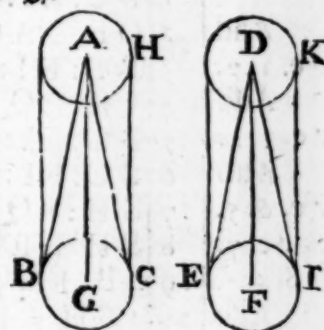
7 $BH = EK$. 9. 5.

8 $\frac{1}{2}BH = \frac{1}{2}EK$. *Ax.* 14.

9 $\frac{1}{2}BH = BAC$, $\frac{1}{2}EK = EDI$. 10.

10 $BAC = EDI$. *Sub.*

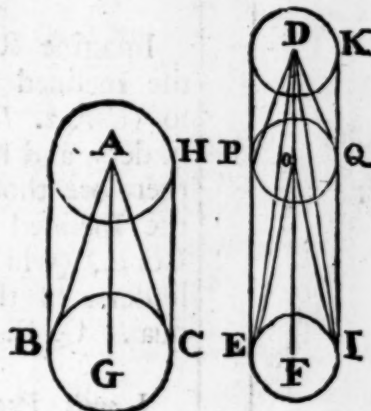
11 $BH = EK$. $BAC = EDI$. Q. E. D.



Case 2 If the Cylinders BH , EK , and Cones BAC , EDI , be Right, and have their Altitudes DF , AG , unequal.

PREPARATION.

From the Greater DF , take $OF = AG$, and thro' O , draw $PQ \parallel EI$.



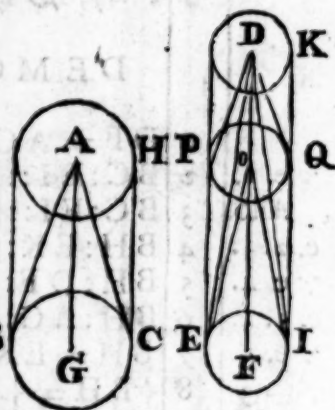
X x x

I say,

I say, $BH = EK$. $BAC = EDI$.

DEMONSTRATION.

- And
 $\therefore$ e. 1. 2. 1 $BC:EI::DF:AG$. Hyp.
 Also 2 $OF = AG$. Const.
 $\therefore$ e. 1. 2. 3 $BC:EI::DF:OF$. Sub.
 Also 4 $EK:EQ::DF:OF$. 14.
 $\therefore$ e. 3. 4. 5 $BC:EI::EK:EQ$. 11. 5.
 Also 6 $BC:EI::BH:EQ$. 11.
 $\therefore$ e. 6. 5. 7 $BH:EQ::EK:EQ$. 11. 5.
 $\therefore$ e. 7. 8 $BH = EK$. 9. 5.
 $8 \div 3$ 9 $\frac{1}{3}BH = \frac{1}{3}EK$. Ax. 14.
 Also 10 $\frac{1}{3}BH = BAC, \frac{1}{3}EK = EDI$. 10. B
 $\therefore$ e. 9. 10. 11 $BAC = EDI$. Sub.
 $\therefore$ e. 8. 11. 12 $BH = EK$. $BAC = EDI$.
 Q. E. D.



Case 3 If the Cylinders and Cones be Inclined, and have their Bases and Altitudes reciprocal, they will be equal.

DEMONSTRATION.

Imagine Right Cylinders to be placed on the Bases of the Inclined Cylinders, of the same Heights with them; so (Cor. 2. Def. 2.) will they be equal to the Inclined Cylinders, and have their Bases and Altitudes reciprocal. Since therefore those Right Cylinders, are (Case 1 and 2.) equal, the Inclined Cylinders, their Equals, as also the Inclined Cones, (which (10.) are the Thirds of the Inclined Cylinders, of the same Bases and Altitudes,) will be also equal. Q. E. D.

I omit Prop. 16 and 17, because too perplexing, and only serving to demonstrate the next, that I shall demonstrate a more easie Way.

PROP.

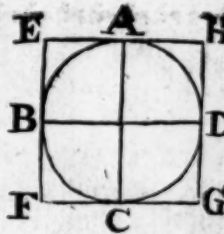
PROP. XVIII.

Spheres ABCD, are in the Triplicate Ratio, or as the Cubes of their Diameters.

PREPARATION.

Let the Spheres ABCD, be inscribed in Cylinders EFGH, whose Altitudes AC, are equal to the Diameters of their Bases FG, as also to the Diameters of the Spheres. Let the Semi-diameters of the Bases of the Cylinders, be (Cor. 2. Lem. 2.) expressed by X and Y, and their Circular Bases by $\frac{1}{2}XZX$ and $\frac{1}{2}YZY$; so (the Diameters of their Bases, or Altitudes, as also the Diameters of the Spheres, being $2X$ and $2Y$.) will the Cylinders be (Cor. 1. Def. 2.) among themselves, as $XXZX$ to $YYZY$, and the Spheres (the Ratio of the Sphere to the Cylinder of the same Diameter and Altitude (whatever it be) being expressed by R) as $RXXZX$ to $RYYZY$. Moreover the Cubes of the Diameters of the Spheres will be $8XXX$ and $8YYY$.

I say, $8X^3 : 8Y^3 :: RXXZX : RYYZY$.

DEMONSTRATION. 

$8X^3 : 8Y^3 :: RXXZX : RYYZY$.
 Lem. 2. 5. Q. E. D.



F I N I S.

И О Т А Л А В Я

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ERRATA.

Page 42. Line 10 and 24. read $\angle AEB$. p. 134. l. 14.
r. $\angle FDB$. p. 151. l. 6. r. $\frac{A}{C} \square \frac{B}{C}$. p. 224. l. 10. r.
 $\Box$. p. 236. l. 14. r. Triangle. p. 260. l. 14. r. $8\frac{1}{2}$.

